MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY.

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Professor A. FOWLER, F.R.S., PRESIDENT, in the Chair.

Leslie John Comrie, M.A., Auckland, New Zealand; temporary address, St. John's College, Cambridge; and
Professor J. S. Plaskett, Director of the Dominion Astronomical Observatory, Victoria, British Columbia,

were balloted for and duly elected Fellows of the Society.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended:—

Christopher Patrick Adamson, Lieut. (late R.A.F.), Corleone, Longham, Wimborne, Dorset (proposed by W. F. Denning);
Clement Osborn Bartrum, B.Sc. Lond., 32 Willoughby Road, Hampstead, N.W. 3 (proposed by Major P. H. Hepburn);
Frederick E. Brasch, Assistant Reference Librarian, John Crerar Library, Chicago, U.S.A. (proposed by F. R. Moulton);
Sidney Bertram Gaythorpe, Engraver and Draughtsman, 3 Prospect Road, Barrow-in-Furness (proposed by Theodore E. R. Phillips);
James Hargreaves, Bennington, Stevenage, Herts (proposed by E. Walter Maunder);
John Johnston Ross, M.A. (Glasgow), B.A. (Lond.), Vice-President, West of Scotland Branch, British Astronomical Association, 7 Queen's Terrace, Glasgow, W. (proposed by Sir F. W. Dyson);
Miss Dorothy Wrinch, Lecturer in Mathematics, University College, London (proposed by J. W. Nicholson).

Fifty-three presents were announced as having been received since the last ordinary meeting, including, amongst others:

Photographs of the Moon, Nebulæ, etc., taken with the 100-in. Reflector of the Mount Wilson Observatory (nine lantern slides and two enlarged prints), presented by Prof. G. E. Hale.

DISCUSSION ON THE THEORY OF RELATIVITY.

The President.—In view of the widespread interest in the theory of relativity which has been aroused by the announcement of the eclipse results, the Council felt that they would be acting in accordance with the wishes of the Fellows if they decided to devote to-day's meeting to the consideration of this important subject. It is scarcely necessary to say that the subject is of special interest to this Society, because astronomical observations provide the principal means—perhaps the only means—by which the theory can be tested. I may remind you that the development of the theory has been closely followed in our publications. There were Council Notes in the Annual Reports for 1910 and 1917, and in the interval we had four valuable papers from our Associate Professor de Sitter, in which the whole mathematical theory was presented. We are asked now completely to revise our conceptions of space and time, and to take a new view of the laws on which astronomy should be properly founded. Such revolutionary changes are not to be undertaken lightly, and I think most of us will be glad of the opportunity presented to-day for reconsidering the evidence which has been put forward in favour of Einstein's theory. Professor Eddington has very kindly consented to open this discussion, and I will ask him to be good enough to deliver his address.

Professor Eddington.—The generalised relativity theory is a most profound theory of Nature, embracing almost all the phenomena of physics, and it is evident that I must devote the time at my disposal to only one aspect of it. I have chosen not to deal with the inadequacy of the old views of space and time which have made a new theory desirable; nor the arguments in support of the relativity theory, whether philosophical or experimental; nor the deductions from the theory, of which the most outstanding is a revised law of gravitation now confirmed by experiment. I propose simply to explain, as far as I can, what are the conceptions of space and time and force which the relativity theory introduces.

The various happenings around us in Nature are located by us in a regular order—an order of space and an order of time. The idea that these two kinds of ordering might be amalgamated, so

that time might be considered as a sort of fourth dimension, is not new; but until recently this was regarded merely as a picturesque way of looking at things, possessing no particular advantages. But the relativity view is that in Nature the combined space-time is fundamental; the distinction of space and time separately is merely an idiosyncrasy of the observer, and indeed observers in different circumstances will not make the same dissection. An observer may distinguish directions in space as right-and-left, backwards-and-forwards, etc.; but we never imagine that there is anything corresponding to this in external Nature. We know that it is simply relative to the way the observer is standing. The new view is that the distinction between here-and-there and sooner-and-later is similarly a matter personal to the observer, depending on the circumstances of his motion. You might compare it in this way. Taking a sheet of paper as representing a two-dimensional continuum, you can pile sheets of paper one on another and build up a three-dimensional continuum—a solid. But if you make a cube in that way it is not quite the same thing as a solid cube of paper. When you examine it, it falls apart into separate sheets. If we similarly take an ordinary solid at different moments, we can regard the taking of it at successive moments as like the piling up of the sheets of paper. But the relativity view is that the true analogy is not the pile of sheets, but the solid block; that is to say, there is no natural division of space and time into which it will fall apart. You can slice it—you can take slices which will be sections in space at different times; but there is no particular direction in which you are bound to perform that operation.

In the four-dimensional world a particle occupies not one point, but a chain of points—its successive positions at successive instants. Hence its history is represented by a line, which is called its world-line. Space-time is filled with world-lines of various objects, like a tangle of threads; but so far there is nothing to discriminate which direction is time and which is space.

Now let an observer appear on the scene. He, too, has his world-line, and naturally for him that is the most important world-line of all. That world-line represents time for him, and he arranges everything else in space and time with reference to it, just as though he were the most important thing in the universe.

So far, then, as there is any distinction between space and time in external Nature it is simply this—that particles of matter are one-dimensional tracks or tubes in a four-dimensional continuum, and the extended dimension is time, the proper-time for the particle. Any one of the tangled tracks can be selected and a consistent division of space and time based on it, and in practice the observer's own track is generally selected.

One slight modification of this statement must be made. The idea of time and space is primarily in the observer's consciousness. I suppose our perception of the flight of time is related to certain molecular processes going on in the brain and body which measure

time like somewhat crude clocks. Space is perceived by vision and by muscular efforts, and those are rough but scientific measures comparable with the triangulations and base-measurements of geodesy. We need not draw any distinction between the space and time directly perceived and that measured by scientific appliances; the observer's brain and body are part of his measuring apparatus, and his clocks and scales merely refine and extend the perceptions of his senses. Now, when an observer makes no conscious effort to change his position in the world, the only change he perceives is a progress through time. By a conscious effort he can make and perceive a progress through space; that is to say, he can walk about. Thus, strictly speaking, the observer's separation of space and time is this—time is that dimension or direction in which we progress without effort on our part; space is the direction in which we progress with effort.

The astronomer naturally raises an objection here. The observer is also borne through space without effort on his part by the earth in its journey round the sun. We must reply to him that his distinction between space and time is a very artificial one. In ordinary life and in the physical laboratory we divide space and time by the rule previously stated, just as though Copernicus had never existed. I take a few steps. How far did I walk? "About five yards" is the reply. If you really believe Copernicus, you ought to reply 100 miles, because that is about the distance from start to finish, allowing for the earth's motion. I have said that any one of the tangle of world-lines might be selected as a basis for separating space and time; and it is therefore legitimate for Copernicus, if he wishes, to select the world-line of the sun for his time-direction. This has certain advantages over the world-line of a terrestrial observer, but it is clearly artificial, because (1) there is in external Nature no separation of space and time at all, and (2) it disagrees with the only natural separation of space and time that exists, namely, that in an observer's consciousness. So that Copernicus' separation is neither one thing nor the other; it is merely a convenient half-way house.

It would, of course, be very inconvenient for observers, when they came to compare notes, if each used the time appropriate to his own course through the world. But this is minimised by the fact that all observers with whom we are acquainted are tied to the earth, and it is unusual to carry out scientific measures in a railway train. The physicist has a rough-and-ready rule for allowing for the difference of course in space-time of two observers, or, as he usually describes it, their relative velocity; so that he can write down almost automatically the translation of the results from one space-time system to another. But the correction usually applied is only approximate.

The change in the system of space and time for observers in relative motion is much more fundamental than that represented by merely subtracting a constant velocity from everything. It means

a real rotation of the time-direction in the four-dimensional world; a bit of space goes into time, and a bit of time into space: and it is these more far-reaching effects which for so long puzzled those who were discussing the result of the Michelson-Morley experiment.

Now, if we view the world of Nature, abstracting everything which is merely contributed by the observer, it is very different from what is generally imagined. Not only is time fused with space, but the most familiar properties of both time and space are gone. In fact it is fair to say that space and time are contributed by the observer; in Nature there is left a certain metrical ordering which is scarcely recognisable as space or time. For one thing there is no such thing as "shape," no difference between straight and crooked, no measure of an angle. We cannot describe this world in familiar language, because we have abstracted from it those conceptions supplied by the observer which might render it familiar.

We have to build up an apparatus for the description of this world from the beginning. An instant of time at a point of space is called an event; so that an event corresponds to a point in the four-dimensional world. Between any two neighbouring events a certain measurable relation exists, called an interval. We can actually measure an interval with material scales and clocks, according to definite rules which I need not describe here. It is possible that we have not yet reached finality as to the right way of measuring an interval, and I ought to mention that it is on this that the question of the displacement of the Fraunhofer lines to the red especially turns. But the important point is that the interval is something inherent in external Nature, and is not like distance and duration, terms which have no meaning until some observer is specified. The interval is the key to all the metrical qualities of the external world.

I have said that there is no distinction between straight and crooked in this four-dimensional world; but it is possible to join two points or events by a certain unique track, which plays a part corresponding to a straight track. Let us take a number of tracks between two points A and B. We divide each track into sufficiently small portions, measure the intervals corresponding to each portion, and sum them. We thus get the interval-length of each track. Now, there is one track from A to B which has the maximum interval-length. This unique track is called a geodesic. It shows how far we have got from the ordinary geometry that the unique track should have maximum length. In ordinary space there is one track of minimum length—the straight line—and any number of longer tracks up to infinite length. But in space-time the longest track is unique, and you can have an infinite variety of shorter tracks down to zero length. Now, these geodesics, besides being the elements from which we build up the geometry of the world, have a special meaning in mechanics. Any particle which is not being disturbed by electric forces or impacts describes a geodesic or longest track. Take, for instance, the earth in a

certain position now and in its actual position a hundred years ago. Those are two definite points or events, A and B, in space-time. What has the earth done meanwhile? It has taken the longest possible track from B to A. In the weird geometry of the space around the sun that longest track is a spiral—imagine a point describing a closed orbit and displacing itself uniformly along a third dimension (time), and you get a spiral. So although a geodesic in some respects plays the part of a straight line, it must not be thought of as looking like a straight line.

This interval-length which we have been using has a fairly simple physical interpretation. It is simply the time as measured by an observer (either by his perceptions or by scientific appliances carried with him) who follows the track.* This is called the proper time and has nothing to do with the time that the independent onlooker introduces to divide up the space-time world. Now, it would have been easy for the earth to get from B to A in no time (as judged by somebody travelling on it). If it went with the speed of light, all the clocks on it would stop and all the physiological processes in the observer would be suspended, because the inertia of everything moving with the velocity of light is infinite. So the earth need only cruise about in the neighbourhood with the velocity of light until the right moment arrived to report itself at A; then it would have done its job in no time—nothing on the earth could have happened in the interval. However, the earth does not do this; it is bound by rules, and the rule of the great trade union of matter is that the longest possible time must be taken over any job. This law of motion is part of the law of gravitation. It is one of the strong points of Einstein's theory which is not generally known, that the laws of motion are consequences of his law of gravitation, not, as in Newton's theory, supplementary laws.

Now let me draw a few world lines (Blackboard). They do not really look like that. They do not look like anything we can picture, because they have no shape—yet. However, this will give us something definite to talk about. We must have an observer. There he is. Hitherto the world was, so to speak, symmetrical, with no distinction of space and time. But the observer is a very unsymmetrical object, tremendously elongated in one direction, and when he contemplates the world he introduces his own asymmetry into it. He is so constituted that he cannot be aware of himself all at once, and his extension is forced on his consciousness as an existence through time. Starting from the time he is immediately aware of, he extends his time-partitions outwards through the world according to the definite rules of interval measurement which he thinks proper. Similarly he draws space partitions—I have only room for one set of them on this two-

* This simple interpretation raises all sorts of difficulties. If the track is not a geodesic, the observer cannot follow it unless he is acted on by electric forces or impacts, and in that case we can scarcely predict how his perceptions of time may be deranged.

dimensional blackboard. He has now located all these world-lines in a space and time; and, for example, this intersection of two world-lines (a collision of two particles) is an event in a particular space-and-time partition—that is to say, it occurs at a definite place and time. Imagine this blackboard to be a sheet of rubber; distort it until these irregular meshes become neat uniform squares, and we shall have a picture of the world as arranged in the space and time of the observer. Things now have a definite shape, length, and duration. But that is only when we have first assigned the world-line of the observer.

A very interesting point arises if the observer's world-line takes a bend—a bend away from a geodesic, because a bend from the straight line has no meaning. Apart from possible electric fields, this can only happen if something hits him and keeps on hitting him. He may recognise that he is disturbed, and deliberately allow for it, choosing to partition space and time according to the experience of a more normal observer—he does not wish to upset the world just because he happens to get into a train. More often, however, he does not notice the blows; he overlooks the fact that the particles of the floor keep on hitting the soles of his boots. And so he has a somewhat illogical prejudice in favour of his own very disturbed track through space as a basis of space-time separation, rather than that of a gentleman who has fallen out of an aeroplane and is temporarily pursuing an undisturbed track.

The result of this deviation from the geodesic is that our observer believes himself surrounded by a field of force. That is merely the way he pictures to himself the altered relation of his space-time partitions to the natural tracks (the geodesics) in the world of Nature. If his deviation from the natural track is due to the ground bombarding his boots, then he says he is in a field of gravitational force; if it is due to the earth whirling him round in its diurnal revolution, he says he is in a field of centrifugal force; if it is due to a ship plunging up and down, then he has yet another name for his symptoms. All these fields of force are essentially of the same nature; they are due to the observer departing from his natural track, but attributing the change to the innocent objects around him.

This view of gravitation is absurdly simple. Suppose an apple becomes loose from a tree, and the result is an impact with the head of someone sitting below. We invoke a mysterious agency called gravitation that is supposed to have acted on the apple and induced it to perpetrate the outrage. But there is a much simpler explanation: the apple is entirely free, suspended in space; but the sitter is being continually bombarded by the molecules of his chair urging him towards the apple. He has only himself to blame as with ever-increasing velocity he goes charging into the innocent apple.

Now, at first there appears an obvious objection. Suppose the same incident is being repeated in New Zealand. According to our views, the two victims are being propelled by the ground with

ever-increasing velocity in opposite directions, and yet they never get any farther apart. Surely that contradicts the laws of geometry. Well, the laws of geometry are what we lay them down to be; and if the laws we use do not agree with the facts of Nature, would it not be a good idea to choose a new geometry which does bear some resemblance to the facts? And so we find that in the gravitational field round the earth, space and time do not fit in with Euclidean geometry, but require a form of non-Euclidean geometry developed by Riemann.

I am trying to show that the thing may be looked at from other points of view. Looking at a thing from a new point of view does not necessarily mean that you have got an essentially new theory in physics. But the point is that in Einstein's theory we focus attention not on the straight line, which merely depends on the particular observer, but on the geodesic, which is a natural track and really has some significance in Nature, and build up our geometry based on geodesics. So that from that point of view it becomes more natural to pay attention to an acceleration relative to the geodesic, instead of relative to the straight line. That brings us to the point at which we have to leave following up this subject without mathematics, because it is clearly a matter of developing the geometry of geodesics. So I cannot trace the steps by which Einstein found that, looked at in this way from the point of view of geodesics instead of straight lines, the Newtonian law of gravitation could not have any exact significance in Nature; but that there was a law expressible in terms of geodesics which was very near to the Newtonian law in the ordinary cases in which we are accustomed to apply the latter. As you know, the two deviations predicted in the case of the passage of light near the sun and the motion of a fast-moving planet, Mercury, do conform to experiment.

There is one thing further which I should like to say, that is with regard to this welding of space and time into a four-dimensional continuum. I think it must seem repugnant to many, since it seems to picture all the events of the world, as it were, mapped out in this four-dimensional space. Events do not happen; we just come across them in the voyage of life. That hardly seems to accord with what we know of time. At least we believe there is a difference between past and future which is quite different from the distinction between left and right. The past is unalterable; but the future we are able to mould. Although I cannot go into it now, that distinction between an absolute past and an absolute future is really preserved in this representation of time as a fourth dimension; it was preserved by Minkowski, and can, if necessary, be emphasised. But we do not emphasise it very much, because it has nothing to do with physics. In physics the past is minus t and the future plus t, just as the left is minus x and the right plus x. Physics ignores free-will,—all its equations are perfectly determinate, and if you have a perfectly determinate scheme I think all the essential difference between space and time vanishes. My point is that although you are not likely to find

much about this distinction of absolute past and future in physics, our four-dimensional world does agree with common experience in preserving a distinction.

The President.—I think we shall all be grateful to Professor Eddington for this masterly exposition of so difficult a subject.

I will next call upon Mr. Jeans. There are certain physical laboratory experiments which have an important bearing upon the theory of relativity, and Mr. Jeans has kindly undertaken to supplement Professor Eddington's remarks by giving us an account of them.

Mr. Jeans.—I would like to congratulate Professor Eddington on the extraordinarily complete and clear view which he has given us of the whole structure of relativity. I am going to attempt a much less ambitious programme; I shall merely try to explain a little how the foundation stones of that structure were laid. I shall approach the question from the point of view of a physicist. The physicists have had the advantage that they were present at the laying of the foundation stones; they discussed the experiments one with another, and were in a position to judge whether the foundation stones were squarely and truly laid.

Let us start by considering what was historically the beginning of the theory of relativity, the question of the propagation of a beam of light. Before the advent of the relativity theory there were two ways in which it had been imagined that light might be propagated, and I am not aware that any beyond those two had ever been seriously considerd by scientists. The first was the undulatory theory, according to which light was simply a system of waves in a medium called ether. The second was the corpuscular or emissive theory, according to which a beam of light consisted in some way of particles projected from the source of light. According to the undulatory theory, a light source at any point P in the ether emitted waves, which, after a certain interval of time, would form a sphere having P as centre. If in the meantime the observer had moved in the ether from the point P to the point O, the wave surface, as observed by that observer, would be an excentric sphere, a sphere of which he was not the centre. In the same way, on the corpuscular theory of light, a source of light might emit waves at P at a certain instant, and afterwards might move to Q. The observer might move to some other point, S. The wave front would have moved with a motion compounded of the velocity of light—*i.e.* the velocity of the corpuscles—and the velocity of the source, so that again we should get a wave front which to any observer would in general be an excentric sphere. I will just write down on the blackboard the equations of those spheres.

$$\text{On undulatory theory}: \quad r = a\left(1 + \frac{u}{V}\cos\phi\right)$$

$$\text{On corpuscular theory}: \quad r = a\left(1 + \frac{u - v}{V}\cos\phi\right)$$

[V = velocity of light, u = velocity of observer, v = velocity of source.]

If either of these theories of the nature of light were true, it would be possible to determine by experiment either u or $u - v$, or possibly both of them. On appealing to experiment it is found that the wave surface has neither of those forms. It has the form $r = a$ simply; in other words, it is always a sphere having the observer as centre. Now, the justification for that statement depends on two experiments. The first is the famous Michelson-Morley experiment. In that experiment a source of light was taken and split into two parts. One part was made to go to a mirror and back in one direction; the other part was made to go to a mirror and back in another direction. The result of the experiment was that the average velocity to and from the mirror was the same in both directions. The apparatus was turned through any angle, and still the same result was obtained. That did not of course justify us in saying that the average velocity in all directions was the same. To complete the experimental evidence, we must fall back on some quite recent experiments of Majorana. Considering a single beam of light which was sent to a mirror and brought back again, Majorana found that whatever the velocity with which the mirror moved relative to the observer or to the source of light, the velocity on the forward path was the same as the velocity on the backward path. In other words, the velocity is not changed by the mirror, no matter what the motion. So, instead of speaking of the average to-and-fro velocity in any direction, we can alter our statement of the Michelson and Morley experiment and simply speak of the velocity in one single direction, because back and forward velocities are the same. The Michelson-Morley experiment, supplemented by Majorana's experiments, now takes the form that the velocity of light in all directions is the same, whatever the motion of the observer. That was originally taken by Einstein as the hypothesis—and it did at the time appear to be a hypothesis—on which he founded his theory of relativity. It has now, by Majorana's experiments, combined with the Michelson-Morley experiment, been raised from the position of a hypothesis to an experimental fact. What we have to do now is not to speak about the hypothesis of relativity, but to examine, as the *theory* of relativity, what are the consequences of this fact—that no matter what the velocity of the observer is, the light surface, as observed by that observer, is invariably a sphere having that observer as centre. According to the older views, a definite point in the ether or a moving source of light was the centre of the sphere. According to the newer views, attested to by the facts of observation, it is the observer himself who is the centre of the sphere. We seem to have here our first glimpse of a more subjective view of the universe, forced on us by the new experimental facts; each observer is, for himself, so to speak, the centre of the universe. He is at rest, and everything else moves past him. Let us now start from this one fact, that for every observer the wave surface is a sphere with that observer as centre. If I may put down some quite simple mathematics, I think I can show you the consequences of that.

One experimenter, S, finds that at a time t after the emission of a light signal, the wave surface is

$$x^2 + y^2 + z^2 - V^2t^2 = 0,$$

the observer himself being the origin. A second observer, S′, moving with a different velocity, observes precisely similar results, except that he finds that he, and not S, is the origin. We assume the wave-surface to have a real objective existence; then, since a sphere cannot have two separate centres simultaneously, we conclude that S and S′ must in some way mean different things when they each assert that they are the centre of the sphere. Put more mathematically, they must be using different systems of co-ordinates in which to record their observations. If S″s co-ordinates are distinguished by accents, his wave-surface will be

$$x'^2 + y'^2 + z'^2 - V^2t'^2 = 0,$$

where we suppose S″s units of measurement to be such that the velocity of light has the same numerical value for him as for S. If we now assume that light appears to both S and S′ to travel in straight lines with uniform velocity, then a linear transformation between x, y, z, t and x', y', z', t' must transform either of the above equations into the other. Put $iVt = \tau$ and $iVt' = \tau'$, and we see that $x^2 + y^2 + z^2 + \tau^2$ must transform into $x'^2 + y'^2 + z'^2 + \tau'^2$. Thus, if a four-dimensional continuum is constructed having x, y, z, τ as orthogonal co-ordinates, then the axes of x', y', z', τ' must form a second set of orthogonal axes in this same continuum. The frame of reference of S′ is accordingly derived from that of S by a pure rigid-body rotation.

Here we have the four-dimensional space-time continuum to which Professor Eddington has already introduced us. The continuum of S is the same as that of S′ or of any other observer; it appears to be a fundamental objective reality. But S and S′ divide it up into time and space in different ways; what S calls pure time or pure space is called a mixture of the two by S′. Thus space and time separately appear to be subjective phenomena: only the product of the two is real.

The equations expressing that the axes of $x', y', z' \tau'$ are derived from those of x, y, z, τ by a pure rotation are precisely the equations of the famous Lorentz-transformation,

$$x' = \beta(x - ut),\ y' = y,\ z' = z,\ t' = \beta\left(t - \frac{xu}{V^2}\right).$$

We may interpret these equations as we will, but I do not think any physicist can doubt that the equations themselves are really true. Out of them has flowed a stream of brilliant predictions and convincing explanations. I will mention only the explanations of the water-tube experiment and of astronomical aberration, the calculation of the dependence of mass on velocity, and of the forces in ponderable media. I mght go further and claim that this last gravitational theory has provided the final test of all; by the

one simple change of non-conformally twisting the four-dimensional continuum and supposing a particle or light-ray to describe a geodesic rather than a straight line (now a meaningless term) Einstein has given an explanation of gravitation which exactly accounts for the motion of Mercury and for the gravitational deflection of light; but I will leave it to others to lay stress on that. In the original simple theory of relativity, we have a theory which is independent of gravitation. What I have tried to bring out is that that simple theory is supported by experiment all along the line, and that the main equations of that theory are the only possible interpretation of the two experiments of Michelson and Morley, and of Majorana.

The Astronomer Royal said.—I wish to remove misconceptions which some Fellows may have with reference to the observations. (Showing on the screen an enlargement from one of the eclipse photographs of the stars κ_1 and κ_2 Tauri, he remarked that the diameters of the images were 4″ and 3″ respectively, while the displacements were quantities of 1″ and less.) In linear measurement the quantities dealt with are of the order of 1/100th of a millimetre. These displacements are small, but Astronomers familiar with stellar photographs will, I am sure, support the statement that quantities of this order are readily measured, if only good photographs are secured.

The President.—I am sure you will all give a hearty welcome to Sir Oliver Lodge. Sir Oliver has informed me that he has not previously attended a meeting of our Society, but you will be interested to know that he also says that he follows our proceedings pretty closely from the reports in a well-known magazine.

Sir Oliver Lodge.—May I thank you, sir, for your welcome? To save time, I may say that I have expressed myself on this relativity theory in one or two places which are very accessible, so that I will try not to repeat anything that I have there said: one is in this month's *Nineteenth Century*; another is in next month's *Fortnightly Review*; and a third is in next Sunday's *Observer*. May I say also that I heartily accept the Astronomer Royal's and the other Astronomers' verdict of the result, and congratulate everyone who has had to do with it? But I am not prepared to accept or to swallow the whole of the theory of time and space which has been put before us this afternoon with so much brilliancy. And one of the things which astonishes me is that Professor Eddington thinks he understands it all, or, I prefer to say, does understand it all. To dispense with a straight line as the shortest distance between two points, and to be satisfied with a crazy geodesic that is the longest distance between two points, is very puzzling. The only analogy I have thought of is the very simple one of the way in which water runs downhill. Anybody would think that a watercourse would take the steepest path; but

it does not. A watercourse takes the most leisurely way, and is always to be found where the contour-lines are farthest apart. (This is only a matter of common sense really, because a river is, after all, a flowing lake, and the water is found where it longest remains.)

The whole relativity trouble arises from giving up the ether as the standard of reference—ignoring absolute motion through the ether,—rejecting the ether as our standard of reference, and replacing it by the observer. By putting the observer in the fore-front and taking him as the standard of reference you get complexity. If you describe a landscape in terms of a man in a train looking out of the window, the description is necessarily complicated. The surprising thing is that this theory has arrived at verifiable results; it is marvellous, and it represents very brilliant mathematical work. It has been done by using an out-of-the-way calculus developed by pure mathematicians—a ponderous kind of tool which only a few people can use. I do not pretend to be able to use it; I only with difficulty follow it; but the principle must coincide with some kind of reality, for by writing down equations on that principle you can get results. The theory is not dynamical. There is no apparent aim at real truth. It is regarded as a convenient mode of expression. Relativists seem just as ready to say you are rising up and hitting the apple as that the apple is falling on you. It is not common sense, but equations can be worked that way.

There was one thing which Mr. Jeans said, or seemed to say, with which I want in a friendly way to disagree. He appeared to say that you can adjust the velocity of light by changing your units. As here stated, that is not true. Changing the units does not affect the velocity of light. Whether you say light travels 186,000 miles a second or whether you say it is so many inches an hour, makes no difference to the velocity. An algebraic symbol ought to represent the thing itself, not a mere number of units. Altering the numerical specification—which is what you do by altering units—makes no difference to the thing itself. Further, I want to ask how can Mr. Jeans say that a sphere of light is concentric with the observer! Consider a source of light and a wave front which it emitted some time ago. Follow that wave front. It is advancing in all directions absolutely concentric with the centre from which it originally emanated. It does not care what the source has done since. The source may have gone away; so may the observer; but that does not matter to the wave front. The light is swimming in stationary ether, like a man who has jumped overboard from a ship and is now swimming through the water. The ship might disappear: it does not matter. Light is concentric with its original source. Now, consider an observer who is going to catch that light, or one who has caught that light. How can the sphere be concentric with either observer? I am surprised that the observer comes in, or that it matters what he is doing. The wave front is quite independent of him. I know that the observer is considered

very important; but there is evidently something here which I do not understand.

Now about the main position. I did not know the way in which Professor Eddington was going to present us with the problem. He presented us with the world-line in geometrical fashion to-day, but previously he has presented us with it in this sort of fashion—

$$ds^2 = -(1-2M/r)^{-1}dr^2 + \text{etc.} + (1-2M/r)dt^2.$$

Those who followed the reports in *The Observatory*, which is the magazine to which the President referred, will remember that sort of equation in polar co-ordinates—the θ and ϕ here omitted. Everything depends on that denominator, the divisor of dr^2, which Einstein and Eddington write as $1-2M/r$. The second term, more clearly written as $\frac{2M/r}{c^2}$, is the gravitational potential at any point divided by the square of the velocity of light. The numerator is the squared velocity of free fall from infinity. And as a beam of light has really fallen from infinity, the expression at once assumes a common-sense aspect.

But, you will say, that is only a corpuscular theory of light. That consideration yields only a Newtonian deflection. The deflection you get from a following particle is only half the right deflection which the equation gives, because in the relations between dr^2 and dt^2 the factor $1-2M/r$ enters twice. But I suggest that if an oblique force is acting on light, as if light had weight, we must consider its two components, a longitudinal component and a transverse component. The transverse component will produce a deflection, just as if it were a particle. The longitudinal component would increase the speed, if it were a particle; but you cannot increase the speed of light. Light is going at its own speed. I agree that it is perfectly constant through the ether: I do not say what happens if the ether could be moved. You cannot expect to increase the speed in a stationary ether by gravitation or anything else. What happens to the longitudinal component? What effect does that produce? Here comes in the hydrodynamic constitution of the ether, the explanation of its elasticity by a fine-grained vorticity, with a dimension which may be of the order 10^{-33} centimetre,—a vorticity which gives it its elastic pseudo-solid rigidity and enables it to propagate light. That vortex motion in the ether, I conceive—but I would ask mathematicians to work it out,—will confer upon the longitudinal component a gyrostatic action, and will double the deflection you would otherwise get. Both the longitudinal and the transverse component thus co-operate, because the longitudinal cannot produce the straight-forward effect which it would like to produce; the deviation is perpendicular to the force, if there is any other rotation at all; and so the deflection is increased by a kind of gyrostatic action—the result of the speed-accelerating component of gravitational force. That is a direct gravitational way of looking at the problem. The optical way is by introducing

the simple idea of refractivity, through a diminution of the velocity of light by a gravitational effect upon the ether's elastic or dielectric coefficient, employing the same factor as expressive of the refractive index. The result comes out in this way all right, if the reduction in light speed is permitted; and no condensation of an incompressible ether is necessary.

In conclusion, Jupiter is just big enough to show a possibly perceptible deflection. If you have two stars, one on each side of Jupiter about two diameters apart, arithmetic shows that their relative shift would be one-sixtieth of a second. Am I right in saying that one-sixtieth of a second is measurable by a heliometer?

The Astronomer Royal.—I think it would be too small to measure not only by a heliometer but even by a photographic plate.

Mr. Jeans.—I would like to ask one question. If, as Sir Oliver Lodge maintains, this sphere is not concentric with the observer, why will Sir Oliver Lodge not devise some experiment which shall reveal to us the distance from the observer to the centre of that sphere? If Sir Oliver Lodge will do that, we shall at once know our velocity through the ether, and Sir Oliver Lodge's belief in the ether will be confirmed.

The President.—Perhaps Sir Oliver Lodge would like time to think that over.

The following communication from Sir Joseph Larmor was then read :—

The Relativity of the Forces of Nature.

1. There was a race of flatworms, known to the geometers, whose activities were confined to a curved region of only two dimensions, but who persisted in assuming that they were really living in a plane. Their scientific surveys of their universe would of course be in discrepancy with the geometry of a plane as formulated by Euclid, particularly on any highly curved hump on the surface on which they lived. But they would not be upset by that: they would assert that there were definite universal forces of Nature in operation, constraining their paths of freely moving bodies and their rays of light to be curved instead of being the straightest or shortest lines on their surface; and they would investigate the theory of these intrinsic forces scientifically. Whence it came about that their philosophers had to propound a theory that every body was attracting every other body in their universe according to a definite law which they assigned. Especially was this conclusion promoted when they came to know that in the terrestrial scene there were forces of gravitation which affect absolutely in the same way all physical lines—that is, those which are paths of actual particles or of rays as contrasted with ideal lines which may be imagined but do not come into being. They would

conclude from analogy that *their* world also was bothered by a universal force of gravitation, whereas their real trouble was that they thought they were living in a flat surface instead of the curved one which was their actual universe.

There was no reason whatever, so far, to compel a choice as to their way of thinking: they may regard themselves as living in a curved surface undisturbed, or else in a flat surface in a world whose phenomena are disturbed by the suitable forces which they can call gravitation. But when they came to look very closely into the scheme which they developed, of the ordering of Nature, on their own notion of a flat universe, there were just a few—really only two—extremely minute features which did not fit in * with their notion of the rest. And when they began to realise that their universe might not be flat, and that there might not be these universal forces of gravitation at all which are only the consequences of their imagining it to be flat, they proceeded to determine their bearings, on this hypothesis, on the curved surface to which it led; when they found that the minute discrepancies inherent in their previous theory of gravitation had vanished along with that theory itself. This adjustment of two very minute misfits was really all they had to go upon; but they had no hesitation in concluding that up till then they had been living under the delusion that the frame of their universe was simpler than the reality. Yet their mental processes had grown habituated to the geometry of the flat; so that although they now recognised the error of their ways, it was too difficult for them to do otherwise than continue to think out things in terms of the flat, with its artificial forces spoiling the innate simplicity of the cosmos for the sake of its frame.

Whence it is perhaps open to philosophers to surmise that something analogous may hold good for intelligences higher in the scale of being, who expatiate in an extension of wider degrees of freedom. They might even, using the symbolic language which is anathema to the downright practical man, feel themselves able to make assertions such as the following.

The free paths of all particles or bodies and the rays of light are in ultimate reality the straightest or most direct paths in a fourfold non-flat section of an underlying flat hyperspace of five dimensions in space and time; but they present themselves in the guise of more complex orbits described in that fourfold section and ascribed to universally acting force, because men, subsisting in this curved section in the hyperspace, like the flatworms in a curved section in ordinary flat space, yet insist on imagining their section to be flat; and other phenomena of nature are disturbed likewise by this illusion.

The historical world-process may as a first approximation be taken as subsisting as one all-embracing configuration spread out in a fourfold continuum of space-time, practically flat within the

* But see papers printed *infra*.

range of our fields of measurement, so that systems enjoy free mobility in it without undergoing change. The situation of the world-process in the continuum is thus of no consequence; and this, when expressed in ordinary language, involves a physical principle called the principle of relativity, so far as it has no reference to gravitation.

The historical world-process is, more precisely, spread out, not in this flat fourfold, but in a somewhat curved fourfold section of a flat fivefold continuum. The situation of this section in the fivefold is of no consequence, any more than is the situation of a given surface in ordinary space; and that is all that is left of the principle of relativity in its original sense,—which is nothing at all when the fourfold is referred to its own internal co-ordinates, unless its curvature is uniform. It is local irregularities in the curvature of this fourfold that, mainly or alone, are taken to constitute the physical universe, as Riemann, and later Clifford, half-seriously speculated. Mankind insist on interpreting their curved fourfold as uniform and flat; but this could not be done with absolute consistency unless it is a special type of curved region developable like a rolled-up sheet of paper. Yet that does not prevent the habit of mind; and the discrepancies thereby introduced in the simple ultimate scheme of Nature assume the guise of effects of new universal disturbing force by which the scheme is strained and complicated, the chief, by far, of which is the force of gravitation.

Dr. Silberstein.—In view of the very limited time I shall touch but briefly upon four points, which seem to be of some interest.

In the first place, I should like to point out the deviations of the observed displacements from radial directions as exhibited by this slide [a slide was shown]. I have calculated the positions of the seven stars and of their resultant observed displacements from the rectangular components quoted by the Astronomer Royal at the joint meeting of November 6, and Professor Porter, having verified the simple calculation, was kind enough to prepare this slide. The full straight lines terminating in the numbered stars are radial, from the black disc's centre, and the arrowed dotted segments represent the observed displacements in size and direction. While I am aware of the difficulty of these measurements and of the circumstance that the direction is a too severe test for them, yet it seemed to me worth while to note these relations without prejudicing their possible interpretation. These displacements are, at any rate, vectors. It seems, therefore, most natural to draw them as such, instead of treating only the seven pairs of their rectangular components. Especially the relativists, who are viewing so critically every reference framework, should recognise that such components, although necessitated by the properties of the measuring comparator, are palpably artificial auxiliaries. Some of the displacements show considerable deviations from the

corresponding radii. Thus, star No. 11 shows a deviation of almost 35″. It is true that this is the most distant of the seven stars ($r = 7{\cdot}9R$). But there is No. 6, with $r = 4{\cdot}35R$, showing the deviation angle 15″ 40′, and No. 2, at the distance $r = 2{\cdot}05R$, whose deviation angle is 8″ 50′. The remaining stars are more satisfactory: Nos. 10, 5, 3, 4 have only 5″·9, 3″·6, 3″·4, and 0″·2, respectively. Professor Porter draws my attention to the fact that the deviations from radial directions have for five stars (3, 4, 6, 10, and 5, situated near the axis) all one sense, and for the remaining two stars (near the sun's equatorial plane) the opposite sense. This circumstance, as well as the deviations themselves, may be due to some accidental errors or of an accessorial nature; but they may as well be "real," *i.e.* essential, and we should perhaps keep an open mind as to the interpretation of these angular deviations and of the whole observed displacements themselves. The astronomers have got an indubitably positive effect, the deflections. But these results of observation may even be a more precious material than they seem. They may have a broader scope than merely to verify Einstein's theory. But time pressing, I will drop this point, adding only that if we were not prejudiced by Einstein's theory we should not say that the figures strongly indicated a radial law of displacement.

My second point is that, even if the displacements were ideally radial, we may have to account for them in some other, non-Einsteinian way. (If the angular deviations are real, this would perhaps point to a refraction by an irregularly distributed medium or substance.) I do not necessarily follow Professor Newall in his suggestion of refraction by an ordinary atmosphere, or a peculiarly electrified one. But, on the other hand, I do not agree with Professor Lindemann's somewhat apodictic rejection of any such explanation, based upon his estimate of the density of a gas required to produce the observed deflection. Professor Lindemann estimated a density of 10^{-9} or 10^{-8} (500,000 km. above the sun's surface) and said that this would be a "palpably impossible" value (*Observatory*, **41**, 323). This does not seem to me so palpable. Professor Lindemann's result does not seem to be convincing, especially as some of his assumptions on the physical properties of the refracting medium are by no means cogent. To say the least, it seems premature to discard every possibility of a non-gravitational refraction as responsible for the observed displacements.

My third point is the obstinately negative result of attempts at discovering the theoretically predicted shift of the lines in the solar spectrum, which I have already emphasised on previous occasions, pointing out the crucial nature of this test. Einstein's whole theory stands or falls with this effect. In fact, Einstein himself says, in *Times* of 28th November 1919: "The great attraction of the theory is its logical consistency." [As a matter of fact, its complete logical consistency seems still questionable, on account of some geometrical and other subtleties. No doubt Einstein wished to express by these words only the fact that his theory

does not contain any new empirical constants.] "If any deduction from it should prove untenable it must be given up. A modification of it seems impossible without destruction of the whole." This was precisely my contention at the joint meeting in November and on previous occasions. I may add that Professor Sommerfeld, an enthusiastic admirer of Einstein's new theory, just writes me the following lines which I translate from his German postcard:—"I fully understand your emphasis laid on the shift (of spectrum lines) towards the red; if this is really absent, no deviations of any number of stars can help Einstein." In fact, provided that the "clock," in the present case the emitting mechanism, *i.e.* the atom of a given element, is the same at the sun's surface as in terrestrial laboratories, the spectrum shift is an unavoidable consequence of Einstein's theory, being given by the coefficient g_{44} of his line-element. One could think of accounting for its being a nil effect by assuming that the emitting atomic system in the sun is slightly different, so as to compensate the appearance of the factor $\sqrt{g_{44}}$. Thus, for instance, if we adhere to the quantum theory of spectra, the period T of a given emitted line is, apart from a purely numerical factor, proportional to h^3/e^4m, where h is Planck's constant (quantum of "action"), e the charge, and m the mass of an electron, more exactly its mass multiplied by the ratio of the mass of the nucleus to the mass of the electron *plus* that of the nucleus. But this does not seem to lead to any satisfactory solution of the difficulty. In fact, using the suffixes e, s for earth and sun, we should have, for the ratio of periods in question:—

$$T_{ee} : T_{se} = \sqrt{g_{44}}\left(\frac{e^4m}{h^3}\right)_s : \left(\frac{e^4m}{h^3}\right)_e,$$

and since there are reasons to assume $e_e = e_s$, and $h_e = h_s$, the ratio of the periods becomes $m_s\sqrt{g_{44}} : m_e$. Thus, if we require $T_{se} = T_{ee}$ (absence of shift) we must have $m_e : m_s = \sqrt{g_{44}}$. Now, the (electromagnetic) mass of the electron being proportional to e^2/c^2 and inversely proportional to the radius of the electron, this requirement will be readily seen not to lead to any plausible assumption. In short, the difficulty could not be overcome without artificiality which would spoil at once the claimed non-empirical nature of Einstein's theory.

My fourth, and last, point relates to a certain strange and by no means necessary geometrical and chronometrical sensationalism based mostly on erroneous or confused reasoning (for which Einstein himself is not responsible), a sensationalism which of late has penetrated even into the daily papers. And with regard to even those puzzling and paradoxically sounding geo- and chrono-metrical results which are legitimately deduced from Einstein's theory, they will cease to astonish us if we remember that Einstein introduces at the very outset the crudest unanalysed concepts of rigid bodies and ordinary clocks. A theory erected on such foundations cannot, manifestly, claim the capacity of revolutionising our usual concepts

of time and space, or even only of shedding some useful critical light upon them. In order to do so, the theory would have to start with non-metrical concepts such as are projective co-ordinates or label-numbers, the more so as Einstein and his exponents do not omit any opportunity to assure us that the only real things are the "coincidences," the intersection points of the world-lines, and not these lines themselves.

Professor Lindemann.—I should first like to thank Dr. Silberstein for calling my attention to a slip, if it really is a slip. I cannot recall exactly what κ represents, so will not enter into the question more closely at present. There is no doubt in my mind, however, that the density gradient necessary to produce the observed deflection is too large to be compatible with the small amount of light that is scattered by the corona. Then Dr. Silberstein emphasised very strongly that no displacement of the spectral lines towards the red as predicted by Einstein had been observed, and he claims that a failure here would upset the theory completely. I do not agree that the absence of this shift has yet been proved. The predicted effect is 0·63 km./sec. St. John finds values ranging from −0·4 km./sec. to +0·6 km./sec., with an average value of 0·0. Graebe and Bachem agree with Schwarzschild in finding a residual positive effect of about 0·3 km./sec. Further, they show that there is reason to believe that these particular lines are unsymmetrical, and that the centre of the absorption line would be displaced about 0·22 km./sec. towards the violet relative to the point of maximum emission through this cause. Whether this is right or not, it seems to me that discrepancies of 1 km./sec., as found by St. John between these lines, show that they are shifted by some unknown cause, and that it is therefore rash to dogmatise at present on the strength of measurements of such a difficult nature.

Sir Oliver Lodge has suggested that the deflection of light might be explained by assuming a change in the effective dielectric constant near a gravitating body. This way of looking at it had occurred to me, and it sounds quite promising at first since it explains not only the deflection but also the shift of the perihelion of Mercury as well as the change of the frequency implied by the shift of the spectral lines, if this exists. The difficulty is that one has in each case to adopt a different constant in the law, giving the dielectric constant as a function of the gravitational field, unless some other effect intervenes.

One point that has apparently been felt as a paradox by many can be simplified. Professor Eddington has stated that a body tends to travel on a geodesic ds, and that this is the longest possible path between two world-points. Since

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2,$$

the statement that ds^2 is a maximum simply means that dt is a minimum for a given path $\sqrt{dx^2 + dy^2 + dz^2}$ in ordinary three-

dimensional space, *i.e.* it is no more than a restatement of the law of the shortest time.

I should like to call attention to another point that has scarcely been touched on to-night, in which I fear Professor Eddington has misled at least two eminent physicists. He states in his admirable report on Relativity that a natural unit of length may be derived from the values of the velocity of light, the absolute gravitational constant, and Planck's element of action. This unit is $4 \cdot 10^{-33}$ cm., and it has been suggested that no smaller unit of length is physically possible, or some similar conception. That this conclusion is unfounded is immediately clear when one works out the corresponding natural unit of mass from the same three units. This is $5 \cdot 37 \times 10^{-5}$ grammes, *i.e.* more than $\frac{1}{20}$ of a milligramme and certainly not the ultimate unit of mass.

The real difficulties of Einstein's theory appear when one comes to consider the boundary conditions. Thus the suggestion that centrifugal force is due to surrounding masses seems to me very unlikely. If there were no masses in the neighbourhood of a rotating body, it would mean that there should be no centrifugal force. This would appear to conflict with the equivalence hypothesis, for unless centrifugal force were affected by acceleration of the whole system one could distinguish between a field of acceleration and of gravitation simply by measuring the centrifugal force of a rotating body. If centrifugal force were present, one would conclude that there were masses in the neighbourhood. If not, one would conclude that they were absent, and know that one could not be in a gravitational field.

Further, there seems to me a considerable difficulty about the conception of a finite universe. If one maintains the interchangeability of space and time as given by the special theory of relativity, our universe can only be the four-dimensional boundary of a five-dimensional sphere or possibly ellipsoid. In this case not only space but time is finite. This means that one should return to the same world point if one waited long enough, which is Zermelo's objection to the second law of thermodynamics in a highly aggravated form, since the time would be extremely short compared to any permissible period of recurrence.

In view of these difficulties, we are perhaps still justified in examining possible physical interpretations of the observed phenomena. One such, which will shortly be published in greater detail, may be of interest. The analogue to Newton's inverse square law in electrodynamics is Coulomb's inverse square law by which two opposite charges, e_1 and e_2, attract each other with a force $\frac{e_1 e_2}{r^2}$. But all the phenomena of electrodynamics show that this law only holds for charges at rest. What the exact law is when they are moving is not easy to say, since in experimental determinations one always has to deal with closed circuits. Thus, a number of different laws which all give the same result when integrated round a complete circuit are possible: for instance,

Weber assumes $\frac{e_1e_2}{r}\left(1+\frac{\dot{r}^2}{c^2}\right)$, and Riemann $\frac{e_1e_2}{r}\left(1+\frac{r^2\dot{\phi}^2}{c^2}+\frac{\dot{r}^2}{c_2}\right)$ for the potential.

If gravitation is of electromagnetic origin, as it must be if all mass is electromagnetic, some such correction term must appear in the gravitational potential. If one assumes the potential to be $\frac{m_1m_2f}{r}\left(1+\frac{r\dot{\phi}^2}{c^2}\right)$ one finds identically the same equation as Einstein for Mercury's orbit, and under certain fairly plausible assumptions for the deflection of light. Major Griffith and I are at present investigating the question more generally, but even as it stands the fact that such a simple and almost necessary correction to Newton's law may explain the facts seems worthy of notice.

Dr. Jeffreys.—It is very important, with regard to these crucial phenomena of the Einstein theory, that we should be quite sure of the positive effects produced, and that there is no other cause that can produce a considerable fraction of them. If there was any other cause that was capable of producing any observable amount, then that would be enough by itself to invalidate the Einstein theory. One such cause that has been suggested pretty frequently is the existence of considerable quantities of matter in the neighbourhood of the sun, in particular the Zodiacal light and the solar corona. Therefore I thought it worth while to work out exactly what effect these two accumulations of matter can produce on the perihelion of Mercury and on the shifting of the star places. You can determine the density perfectly from the luminosity. You know the intensity of light that falls on these masses; you know the distance; you have a fair idea of the physical constitution; and you simply need compare these things with the light that you receive from the sky at midday to get a direct estimate of the amount of these two deflections that can be got from this mass which is known to exist. The results are rather startling. There is enough matter in the Zodiacal light to account for one three-thousandth of the motion of the perihelion of Mercury. There is enough matter in the corona to account for about 10^{-8} of the motion of perihelion of Mercury. The solar corona, by means of its refraction, is capable of producing one-millionth of the observed deflection of the star images. So I think it is clear that the only reasonable interpretation one can make of these two shifts is that they are due to some property of the law of gravitation. Now, there are three constants in the formula for ds^2 that would be capable, if they exist, of producing observable effects. It so happens that the three known facts, the truth of Kepler's third law, the motion of the perihelion of Mercury, and the displacement of star images, give different equations for the constants, and the only solution that satisfies those three conditions happens to be Einstein's theory. So that Einstein's theory, I think, must be accepted as the only theory that will satisfactorily co-ordinate these facts. On the other hand, there is the phenomenon

said to be predicted by Einstein's theory of the displacement of the spectral lines, and it is claimed that this has not been verified. The spectral shift is not strictly necessary on Einstein's theory; but I have looked at several papers where measurements of this spectral shift have been recorded, and will just write a few of them on the blackboard:—

	Angströms.
Einstein's theoretical shift	0·008
Evershed, from iron lines	0·013
Evershed, from cyanogen	0·008
St. John, from cyanogen	0·002
St. John, from iron lines . (two groups)	0·007
	0·008

These refer to differences between the limb of the sun and the arc, so that the displacements are less likely to be caused by motion in the line of sight at the solar surface. While they do not show any uniform accordance with the theory, they are at least sufficiently in agreement with it to suggest that it cannot be condemned on the spectroscopic evidence alone.

The President.—May I say that I think the case for the spectrum shift has not been perfectly presented either by Professor Lindemann or by Dr. Jeffreys? This matter is not by any means so simple as might appear, but there is no time to go into it now.

Professor Eddington.—I should just like to refer to one point which was raised by Professor Lindemann, about the four-dimensional space-time closing up, and time coming back round the circle. It depends on the sign of the curvature. If you have the curvature negative, then you can compare your world to the sheets of a hyperboloid, one direction going off to infinity and the other directions being more or less closed. You are quite at liberty in your choice of time, and any direction which goes off to infinity at each end may be taken as your time-direction; and the directions in which the sections are comparable to a circle rather than a hyperbola correspond to possible space-directions. So that neither Einstein's nor de Sitter's theory of curved space-time involves real time coming round again. Also he suggested that if you had sufficient matter, as each part was curved, the whole world would close up. One of the difficulties found by Einstein and de Sitter was that you could not manage to make the space close up by matter alone. You had to make some modification of the law of gravitation to make it do so. I think that we are often deceived by the analogy of a three-dimensional surface. When you go beyond three dimensions you get other possibilities coming in, which we do not see in three dimensions. That is one reason why I do not care very much to use that analogy of curvature of space. I think it is very liable to be deceptive; certainly it has often deceived me.

The President.—I am perhaps fortunate in having no time left for myself. I think, however, we may take a reasonable pride in the contribution which our Society has made to the development of this subject through its representatives on the Eclipse Committee, and we may well hope that there will be general satisfaction in the knowledge that our national prejudice did not prevent us from doing anything that we could to forward the progress of science. The new theory is revolutionary in character, but Einstein himself has assured us that Newton's great achievement can never be overthrown in any real sense either by this or any other theory. It is only in a few exceptional cases that the difference between the two laws of gravitation becomes at all appreciable. My principal duty, however, is to ask you to give your very cordial thanks to Professor Eddington, Mr. Jeans, Sir Oliver Lodge, and to the other speakers who have contributed to the discussion.

The vote of thanks was carried with acclamation, and the meeting terminated.

The Relativity of the Forces of Nature. II.

By Sir Joseph Larmor.

It remains to try to justify, so far as may be possible, this allegorical sketch.* The chief footnotes have been added later.†

Its more elementary range of ideas has to be contrasted, as regards structure and results, with the complex and doubtless wider form of development necessitated by the new relativity formulations. A chief contrast rests in the reluctance of the latter to submit to the possibility of any organic standard, to which temporal and special relations can be referred as a whole; instead of a simple underlying frame of reference consisting of a flat or Euclidean space and time, in which any material measuring-

* See *supra*, Discussion on Relativity, p. 109.

† The main question in this discussion is that of the validity of the two predicted effects on waves of light. This is largely a separate matter of interpretation, outside the complex details of the gravitation theory. My imperfect knowledge of the latter will be only too obvious: it is of very recent date, and it may be an excuse that other duties pressed, while even now much of the very extensive relevant literature has not become accessible in this country. It is only since this paper was finished that I have had the opportunity of comparing with Professor Eddington's admirable and instructive *Report*, which I had mislaid; this is probably an advantage from the critical point of view. It may now be permitted to state that I find nothing in his deduction of the effects on light (pp. 47, 53–58) that has as yet shaken my negative conclusions. The question at issue seems to reduce to this: whether it is safe to mix up a strict mathematical analysis of the modified space-time with an extraneous semi-metaphysical and loosely expressed "principle of equivalence," except in so far as verification of that principle is established as part of the analysis. *Cf.* also *Proc. Roy. Soc.*, Nov. 1919; *Proc. Cambridge Phil. Soc.*, Jan. 1920.

systems can be mobile without change of structure, it has to contemplate wandering observers carrying infinitesimal measuring-rods (as with Riemann) and also time-pieces, whose lengths and rates are absolute in some theories, and in others change as they are moved about, in a manner which must be either unconditioned or else be the expression of physical law. The question appears to be insistent, whether except as regards algebraic analysis there is much advantage for any attainable concepts of relativity in this intricate and indirect mode of description of spacial extension.

Co-variance of relations for transformation from one set of internal co-ordinates to others is merely a test that the material system exists as specified in the particular scheme of space and time to which such sets of internal co-ordinates belong: it cannot be a physical criterion discriminating between one constitution of a material system and another.

But, on the other hand, to insist, through illusion or by intention, that the co-ordinates which are its quantitative marks of position with regard to one scheme of space and time are to be henceforth arbitrarily considered as its marks in a different frame of space and time is virtually to replace the material system by a new one.* Though the co-ordinates themselves correspond, their higher spacial and temporal gradients will have different analytical expressions in the new frame; the fundamental variational relation of dynamical science will work out so as to involve new terms in the equations resulting from it; and these terms would be interpreted as the expression of new universal forces in nature. Whereas in the other aspect of the process they are the expression of the misfit that arises from wrongly transferring the material system into an unsuitable frame of space and time.

The universal forces of nature would thus be merely relative, in so far that they undergo change whenever the same relations are fitted into another but incongruent frame of space and time; and the question arises, with Einstein, whether it is possible to arrange in this way that those forces, which arise only as a convenient cause, of universal type, assigned in order to identify certain changes of apparent motion which always appear under similar circumstances, shall disappear altogether by suitable choice of a frame, their potential being thus absorbed into deformation, actually slight though complex, of the previous frame of reference. For instance, a change in the scale of time, varying from place to place, will alter the measures of all accelerations in the system.

* But a correspondence of this kind can be fruitful only in other types of problems; for $\delta\sigma$ will not usually be an element of length in the new scheme x', y', z', t'. If it is to express a relation of one coherent space of reference with another, that must be assured. It is to be done by making the element of extension of the new space not merely correspond point for point to the original, but to be also superposable on it in the Euclidean manner. This involves a scale of length transferable (in an auxiliary space of higher dimensions, from the one element to the other, in fact applicable everywhere and in all directions. The Riemann specification is thus introduced, and the new space must be curved or warped essentially, in the sense of his analysis, from the original flat. *See* Appendix.

In any theory of transformation or correspondence such as is here outlined, all periods of time measured at the same place would be affected in the same way: thus, so far as such a theory can subsist, spectroscopic analysis should show no difference between the lines of hydrogen in light which has come from the sun and in the light of terrestrial hydrogen.

The alternative conclusion would seem to imply that the waves of light act as messengers carrying the scale of local time belonging to their place of origin to all parts of the universe. That would appear to involve the direct establishment of a Newtonian absolute time throughout the cosmos, which is a negation of all the modern assertions of relativity. Cf. *Proc. Camb. Phil. Soc.*, Jan. 1920.

The main difficulty here is that the thread of transition from gravitation theory to light-waves is so slight, the speed of rays being merely assumed abstractly to be a modulus of the spacial extension. In absence of further guidance, if we adhere to correspondence of the gravitational space-time with an underlying simpler one, elements of time are altered at each place in a specified ratio, also elements of length in ratio determined by their directions, and all local velocities, *including ray-velocity*, in the ratio of the quotients. As then the scale of apparent time changes from place to place, the path of a ray from point to point can only be determined by making the number of intermediate wave-lengths minimum—that is, by an equation $\delta\int ds/\lambda = 0$. But if there is a local scale of length, however complex, $\delta s/\lambda$ will be unvarying. Thus, on this view, rays would not be deflected by gravitation at all.

Can Gravitation be absorbed into a Scheme of Space and Time?

In order to examine this question in an elementary way, let us begin with the succinct general mode of expression of orbits in a field of force, in ordinary space and time, so as to be in position to institute comparisons. In any field of force whatever, of potential energy $-mV$, whether it be gravitational or not, the orbit of any particle or body whatever, of mass m, is determined by the Lagrange-Hamilton variational equation

$$\delta\int\left[\tfrac{1}{2}m\left\{\left(\frac{\partial x}{\partial t}\right)^2+\left(\frac{\partial y}{\partial t}\right)^2+\left(\frac{\partial z}{\partial t}\right)^2\right\}+mV\right]dt=0,$$

with fixed limits of time, from which m disappears. This is equivalent approximately to an equation of fourfold spacial type

$$\delta\int\left\{\left(\frac{\partial x}{\partial t}\right)^2+\left(\frac{\partial y}{\partial t}\right)^2+\left(\frac{\partial z}{\partial t}\right)^2-c^2\left(1-\frac{2}{c^2}V\right)\right\}^{\frac{1}{2}}dt=0.$$

For on expanding the latter integrand in descending powers of c it becomes, leaving out a constant factor ic,

$$1-\frac{1}{2c^2}\left\{\left(\frac{\partial x}{\partial t}\right)^2+\left(\frac{\partial y}{\partial t}\right)^2+\left(\frac{\partial z}{\partial t}\right)^2+2V\right\}+\ldots,$$

where the terms omitted involve the square of V/c^2 and $(\text{vel.}/c)^2$: thus, on neglecting this fourth order of small quantities, these two forms of variational equations become the same.

Now the second form of the variational equation is

$$\delta \int d\sigma = 0,$$

where

$$\delta\sigma^2 = \delta x^2 + \delta y^2 + \delta z^2 - c'^2 \delta t^2, \qquad \frac{c'^2}{c^2} = 1 - 2\frac{V}{c^2};$$

that is, it determines the most direct (shortest) lines in a curved fourfold region in which the element of length * is expressed by this form of $\delta\sigma^2$, as contrasted with the corresponding straight lines in the uniform region of Minkowski's form with constant c. We have therefore the result that the free orbits of bodies in a field of force of potential energy $-V$ per unit mass in ordinary space are the same as the direct or straightest paths (geodesics) under no force in such a modified Minkowski fourfold, in which the space still remains Euclidean, but the scale of measure of time varies from point to point in the ratio c/c'. These orbits have, it is true, been in the comparison located in ordinary Newtonian space and time: but the difference between the Newtonian frame and the Minkowski fourfold is for this purpose extremely slight.† In fact if c were infinitely great, any finite changes in space would be associated with infinitesimal changes in time, so that space and time would be independent as with Newton. It is the large variable part of c'^2, equal to $c^2 - 2V$, which allows the absorption of finite gravitation in the curvature of the fourfold.

A remark seems to be necessary here as to the distinctive character of the geometry of a Riemann curved manifold. To fix ideas, let us confine attention to a plane space specified by two Cartesian co-ordinates x, y: any configurations in the plane can be transferred to new co-ordinates x', y', determined by relations

$$x' = f_1(x, y), \qquad y' = f_2(x, y):$$

then the characteristic equation of the line element

$$\delta s^2 = \delta x^2 + \delta y^2$$

assumes a new form,

$$\delta s^2 = f\delta x'^2 + 2g\delta x'\delta y' + h\delta y'^2,$$

in which f, g, h, are functions of x', y'. This form thus still expresses the geometry of the plane. But in the Riemann theory it belongs to geometry on a definite curved surface, and not on a

* The sign here chosen throughout for $\delta\sigma^2$ happens to be the one that lays stress on its spacial features.

† It is, however, of the same order as the Einstein modification of gravitation; which will affect some statements *infra*. It has, I find, been discussed on the older uniform relativity by Professor de Sitter, *M.N.*, 71, 388 (1911), with references, leading to a perihelion advance for Mercury of about 7″.

plane: the feature thus limiting it is the subtle postulate that "lines have a length independent of position and consequently that every line can be measured by every other." This is the equivalent of the transferable measuring-rods of the Einstein theory: though algebraically self-sufficing by virtue of the variational calculus, its significance can hardly be fully grasped except by immersing the curved space under consideration in a flat one of higher dimensions, in which all material bodies, including measuring systems, are freely mobile without strain or other change. The fourfold varying space-time frame of gravitational theory will therefore for purposes of description be conceived as a construct in a uniform fivefold frame, after the manner of a twofold surface in threefold ordinary space.

The Special Postulate of the Einstein Gravitation Theory.

It would appear, therefore, that by this mode of correspondence we could absorb into the space-time scheme any set of universal forces of nature, amenable to an energy function, which we chose to assume. Can there be a further restriction introduced somewhere in the special relativity analysis the effect of which is to lead to the Laplace-Poisson characteristic equation, and so confine the possibilities of choice to the forces of gravitation? At first glance it would seem natural to find this restriction in a postulate introduced apparently for reasons of simplification alone, in an early stage of the analysis. The restriction is imposed that in the correlation between the supposed actual curved fourfold and its assumed flat representation or map, corresponding elements of extension (fourfold volume) are to be equal. Is it this postulate that restricts the universal forces of nature to the one type of gravitation? For it is certainly this restriction that involves the modification of the law of action of that force, which would otherwise be purely Newtonian as above. As this postulate seems to be the fundamental feature,* it becomes desirable to quote, in free translation, the reason assigned for its imposition.

"As therefore the function of position $-g$ is always positive and finite, it comes to hand to postulate a choice of co-ordinates *à posteriori* so that this quantity shall be made equal to unity. We shall see later that by the introduction of such a restriction of choice of co-ordinates we can aim towards notable simplification of the laws of nature." †

* Since this paper was completed, a bright critical appreciation of the Einstein gravitation has become accessible, by F. Kottler, *Ann. der Physik*, 56, 1918, pp. 401 462, in which an essential position appears to be asserted for this postulate, as taking the place of the postulated constancy of c in a non-gravitational field.

† A. Einstein, "Die Grundlage . . ." *Ann. der Physik*, 49, 1916, p. 789, which is the latest statement of the theory that is at present here accessible. [But the series of valuable papers by Professor de Sitter in *M. N. Roy. Astron. Soc.* ought not to have been overlooked, though they do not bear on this question.]

One may perhaps expand this into a statement that in a universe of very high velocities, far beyond our experience in the main, but just grazed in a few minute astronomical particulars, in which mass has become variable with speed and has had to be transcended into relations of transfer of momentum and energy which have to be measured in other ways, this postulate allows of the construction of a scheme of equations of such transfer, of complexity somewhat less than hopeless, which at lower speeds becomes continuous with the manageable laws of flux of energy and momentum in the usual electrodynamics. The one most exceptional and striking case in which such high speeds are actually cognisable for electrodynamics is that of the unknown microscopic system which constitutes a flying electron: more remarkable than the recent astronomical agreements is the fact that, apparently up to four-fifths of the speed of light, the much-altered measure of the mass or steady inertia of an electron, as computed from this general type of theory of correspondence in the non-gravitational field in which absolute velocities are absorbed, is confirmed by experiment.

The restriction to equality of extent is not introduced for the sake of gravitation: that has been covered already, practically in strict agreement* with the Newtonian law which the new postulate complicates. The only reason physically assignable (other than facility of extension into a consistent theory for the complexities of speeds far beyond our range of experience as above) must be that it simplifies and promotes the inclusion in the scheme of the other main features of physical nature, the modes of operation of radiation and electricity, in some way perhaps capable of expression in direct physical terms.

The Mode in which essentially the Einstein Modifications arise.

The theory as developed is thus a theory of absorption of gravitation or other apparent universal forces of nature by change into a new and necessarily complex space-time scheme of reference. But a postulate unnecessary for that purpose is imposed, that in the correspondence between the new (supposed actual or rather appropriate) curved space-time fourfold and the standard flat one, into which the relations expressing the phenomena of nature are forced, the local measures of extension (fourfold volume) are not changed. This postulate does not arise at all from any necessities of this mode of representation of gravitation: it must be something imposed on gravitation to satisfy outside correlative needs, and it in fact requires a very slight alteration of the mode of operation of that supposititious agent, by way of complication, such as can account for the outstanding astronomical discrepancy as regards the movement of the perihelion of the innost planet Mercury. For problems of a spherically symmetrical field the

* But see footnote *supra*.

modification thus imposed is simple: it must change the characteristic equation of the modified space to the polar form

$$\delta\sigma^2 = (1 + 2c^{-2}V)\delta r^2 + (r\delta\theta)^2 + (r\sin\theta\delta\phi)^2 - c^2(1 - 2c^{-2}V)\delta t^2;$$

for only by introducing this new multiplier of δr^2, implying a symmetrical shrinkage of the standard space towards the centre, imparting to it elliptic quality, is the extent of the fourfold element in any locality with this new scale of time preserved the same as it was in the corresponding standard element. Multipliers for the two transverse terms are excluded, because a spherical surface would then be modified into a surface which would not close.* It is now an ordinary problem in the dynamics of orbits to verify that $\delta\int\delta\sigma = 0$ involves gravitation modified so as to give the result stated as regards the perihelia.

Nor is the special postulate of equality of fourfold extents compelled for the treatment of deflection of optical rays. The deduction of an influence on the paths of rays of light does not involve any theory of the structure of radiation, electrodynamic or other. It bases itself immediately on Huygens' principle for undulatory propagation, which asserts that in a medium in which the velocity of light c' varies from place to place, the rays are paths of quickest propagation, for which therefore, $\delta\int c'^{-1}ds = 0$. The change of space-time which absorbs gravitation without any modification of its mode of operation has been shown above to be determined simply by making

$$\frac{1}{c'} = \frac{1}{c}\left(1 + \frac{V}{c^2}\right).$$

Thus the ray-path is expressible by an equation

$$\delta\int v\,ds = 0 \qquad \text{where} \qquad v = \frac{k}{c'} = \frac{k}{c}\left(1 + \frac{V}{c^2}\right),$$

* This reason is wrong. They *could* be modified by a change of co-ordinates, for example, $r = r_1(1 - k.2c^{-2}V)$. Then the question arises, which form of co-ordinates is to be brought into comparison with the polar Newtonian co-ordinates of practical astronomy? Fortunately, in the three Einstein tests it happens to make no difference.

The ratio of corresponding elementary domains can always be made unity by changing the co-ordinates in one of them. For when the expression for $\delta\sigma^2$ is altered by changing to new co-ordinates by an explicit transformation, the new form still represents the same space, and the new co-ordinates the same point in that space, but the element of apparent extent determined from them has been changed. It is when this ratio of extent is made unity for two systems of co-ordinates such as correspond as closely as possible in the two states, that the equality becomes an imposed condition and not a mere adjustment: for example, in the radial field this radial symmetry of $\delta\sigma^2$ is to be retained. On the present point of view, to make it unity artificially by changing one set of the co-ordinates at every place would only be a complication, by spoiling the correspondence of the theories that are to be compared.

or say

$$\tfrac{1}{2}mv^2 - \frac{mk^2\mathrm{V}}{c^4} = \tfrac{1}{2}\frac{mk^2}{c^2};$$

thus the ray is the same as the free orbit of a particle starting from an infinite distance with velocity k/c in a field of potential energy $-mk^2\mathrm{V}/c^4$; or, putting k equal to c^2, it is the same as the orbit in a field of potential energy $-m\mathrm{V}$, with velocity at infinity equal to c.

The path of a ray is therefore, on this interpretation of the unaltered Newtonian law, the same as the free orbit of a particle moving in the field of force: and this applies more generally to a field of any potential energy $-\mathrm{V}$ whatever, whether it be gravitation or any other possible type of universal force. The gravitational deflection of the rays, in a Newtonian space, would thus have half the final Einstein value.

If, further, the restriction to conservation of the element of extension in transferring to the new fourfold is required, which is in its origin extraneous to this mode of representing gravitation taken by itself, the resulting radial shrinkage of the space superposes an additional curvature on a ray passing transversely. The Huygenian principle of constant phase along the wave-front makes the curvature of a ray equal to the gradient of the logarithm of the index in direction transverse to the ray, thus equal here practically to the gradient of c/c' along the radial direction: and if the radial element of length is shrunk, this gradient is increased proportionally. It is easy to recognise * at once that this additional change in space, though of a secondary order, just doubles the previous result belonging to the change in the scale of time required for the absorption of unmodified gravitation, which made the deflection of a ray equal to that in an orbit for a particle coming on with velocity c.

This doubled result was the one finally predicted by Einstein, and is now confirmed by the recent eclipse photographs. The argument has been here analysed, in its simplest approximate elements, in order to support the assertion that it does not involve in any direct way a physical theory of light; it probably does involve such theory in some underlying manner, in the imposed postulate of conservation of element of extent in transfer of the physical system from one space-time fourfold into another.

The Vector Algebra for an Electrodynamic Fivefold.

As our modes of comparison of fourfolds employ for the sake of directness of view an auxiliary uniform and flat fivefold, the

* Care is required. The $\delta\mu/\delta n$ rule is not valid unless μ is isotropic: and its application by Einstein (*Ann. der Physik*, **49**, 1916, p. 822) seems to obtain the desired result by two compensating mistakes. It is applicable to the isotropic co-ordinates ($k=1$) used by Eddington (*Report*, p. 54). But if the observer were near the Sun, say on Mercury instead of the Earth, these two systems of co-ordinates would give different values for the deflection of the ray; and the question would arise, which of them is to be taken as corresponding to the co-ordinates of practical astronomy?

electrodynamic vector scheme for the flat fivefold $xyz\xi t$ should perhaps be evolved and examined.* Employing the vector algebra of Clifford in five dimensions, operating on a single fivefold vector potential U, we have immediately,† c being supposed to be absorbed in the units of measurement and recoverable at the end of the analysis, and the five unit vectors $ijkl\omega$ being subject to the scheme of symbolical relations of type $ij = -ji$, $\iota^2 = -1$,

$$\nabla_1 = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} + l\frac{\partial}{\partial \xi} + o\iota\frac{\partial}{\partial t}$$

$$U = iF + jG + kH + lK + o\iota V, \qquad \iota = \sqrt{-1};$$

thus

$$\nabla_1 U = jk\alpha + ki\beta + ij\gamma + il\alpha' + jl\beta' + kl\gamma' + io\iota P + jo\iota Q + ko\iota R + lo\iota T,$$

in which the expressions for the components α, α', P, . . . need not be here set down, provided

$$\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} + \frac{\partial K}{\partial \xi} - \frac{\partial V}{\partial t} = 0;$$

it is thus a vector of ten components, six of them, $(\alpha\beta\gamma)$ and (PQR), representing the usual magnetic and electric forces. Further

$$\nabla_1^2 U = i\left(\frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z} - \frac{\partial \alpha'}{\partial \xi} - \frac{\partial P}{\partial t}\right) + j(\ \ldots\ ) + k(\ \ldots\ ) - l\left(\frac{\partial \alpha'}{\partial x} + \frac{\partial \beta'}{\partial y} + \frac{\partial \gamma'}{\partial z} - \frac{\partial T}{\partial t}\right)$$

$$- o\iota\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} + \frac{\partial T}{\partial t}\right) + ijk\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial z}\right)$$

$$+ jko\iota\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} + \frac{\partial \alpha}{\partial t}\right) + kio\iota(\ \ldots\ ) + ijo\iota(\ \ldots\ )$$

$$+ ilo\iota\left(\frac{\partial T}{\partial x} - \frac{\partial P}{\partial \xi} + \frac{\partial \alpha'}{\partial t}\right) + jlo\iota(\ \ldots\ ) + klo\iota(\ \ldots\ ).$$

The relation $\nabla_1^2 U = 0$ thus involves twelve algebraic equations, of which ten are independent, among ten independent variables. All but the first four are, however, implied in the expressions, not here set out explicitly, for the components of the vectors involved

* Every curved fourfold that can exist in a flat fivefold is obtained by making ξ some function of x, y, z, t: thus not every fourfold can be so contained, though it can always be embedded in a higher flat manifold. But the space-time of the radial gravitational field, which is the practical case, is obviously of this special type.

† This is the procedure of a cognate paper "On Generalised Relativity," *Proc. Roy. Soc.*, Nov. 1919.

in them in terms of the components of U, as might have been anticipated since ∇_1^2 is a scalar. If, with a view to the particular type of solution belonging to a fourfold, we make α', β', γ', T each identically null, there remain six independent equations appropriate to determine the other six variables, which are the standard circuital relations of ordinary electrodynamics; but, in addition, the gradients along $\delta\xi$ of P, Q, R must vanish.

At first glance the latter conditions might appear to be automatically satisfied on the ground that (PQR) and $(\alpha\beta\gamma)$, determined from the other group of separate equations, do not involve ξ. But that would hold good only in the flat: owing to the curvature the other dimension ξ must come in, unless it is eliminated in the Gauss-Riemann manner by making $\delta\sigma^2$ a general quadratic function of co-ordinates confined to the fourfold. The scheme of equations above will remain invariant on transformation, and so be independent of the frame of co-ordinates, within any element of extension in the fourfold, provided δx, δy, δz, δt represent orthogonal linear elements in it for which $\delta\sigma^2$ reduces to the form $\delta x^2 + \delta y^2 + \delta z^2 - \delta t^2$; but this scheme of relations in space would change from element to element in a curved fourfold.

An exact mode of procedure to avoid such complication would be, now confining to the curved fourfold of reference and writing,

$$(\mathrm{P},\ \mathrm{Q},\ \mathrm{R}) = \frac{\partial}{\partial t}(\mathrm{L},\ \mathrm{M},\ \mathrm{N}),$$

to take (L, M, N) as the independent physical variables, *assuming*, as suggested above, that

$$\alpha = \frac{\partial \mathrm{N}}{\partial y} - \frac{\partial \mathrm{M}}{\partial z},\ \beta = \ \ldots,\ \gamma = \ \ldots$$

The variational electrodynamic equation,* necessarily invariant, as is readily verified, would then be

$$\delta\int(\mathrm{T} - \mathrm{W})d\tau = 0, \qquad d\tau = dxdydzdt,$$

where, c^2 being absorbed as above in units of measurement,

$$8\pi\mathrm{T} = \left(\frac{\partial \mathrm{N}}{\partial y} - \frac{\partial \mathrm{M}}{\partial z}\right)^2 + \ \ldots\ + \ \ldots,$$

$$8\pi\mathrm{W} = \left(\frac{\partial \mathrm{L}}{\partial t}\right)^2 + \left(\frac{\partial \mathrm{M}}{\partial t}\right)^2 + \left(\frac{\partial \mathrm{N}}{\partial t}\right)^2.$$

The variational equation formulated in terms of (L, M, N) and Cartesian elements of length, involving thus a relation of curl, would then be expressed in the Gaussian curvilinear co-ordinates appropriate to the curved fourfold as determined by the expression for $\delta\sigma^2$, and on effecting the variations the equations of electro-

* This is closely cognate to the procedure in *Æther and Matter*, §§ 56–60.

dynamics, as modified for the curvature of the fourfold in which the system is now located, would ensue.

But though this would be a consistent scheme of vectors of the electric and magnetic forces in the electrodynamic field, it would not be consistent with their being derived as a gradient from a vector potential.

An alternative mode of procedure would express the equation of Action in terms of the components of the potential as independent variables, when the variation would determine the mode of their propagation, and thence that of the forces.

For a general form of $\delta\sigma^2$ the principle of Action is not available to secure invariance:* and Einstein makes connection (*Ann. der Physik*, **49**, 1916, p. 812) between the theory of radiation and the theory of gravitation through the vector potential and the tensor scheme. In the present special case of scale of time variable with position, would the two methods agree in result? Is the problem indeterminate?

We conclude that in the more general electrodynamic scheme belonging to a fivefold flat frame the scheme for the fourfold frame cannot exist exactly as a particular case,† except under special conditions. The direct transference of the electric and magnetic forces from their curved fourfold into a misfitting flat one, which seems to be equivalent to the procedure of Einstein (§ 20), implies, if it is interpretable physically, that these vectors are imagined as physical entities in the same way as matter.

Simple Vector Synthesis in the varying Fourfold.

The element of the region specified by the formula for $\delta\sigma'$, in the transformation of the Newtonian gravitation, proves to correspond exactly to the element determined by the form of $\delta\sigma$, except that the scale of times is altered in the ratio c/c' which varies from one point xyz to another: as regards geometrical relations in xyz the elements of fourfold extent are identical. The vector analysis of Clifford, with the automatic security it gives for invariance when co-ordinates are changed, is therefore appropriate to the events arising within a single element, for being infinitesimal the element can be treated as flat: its distance vector R is expressed by

$$\delta\mathrm{R} = i\delta x + j\delta y + k\delta z + o\delta(ic't).$$

The vector calculus applies completely within each element, where there is invariance for all transformations of the 4-hedral frame of reference $ijko$; but for a wider finite region no gradients of vectors beyond the first may be derived by ordinary differentia-

* It has been introduced by Hilbert, followed by Lorentz and de Donder, on wider lines. In the hands of the latter especially, Least Action involves an explicit stress-tensor scheme such as Einstein employed as an exploring principle.

† A hasty presumption to the contrary in the paper previously referred to "On Generalised Relativity," *Proc. Roy. Soc.*, Nov. 1919, p. 353, was corrected in a footnote.

tion, as the higher ones would involve the curvature of the fourfold frame. Any schemes to be formulated on this method must thus now be expressed in terms of differential equations of the first order alone, like the Maxwellian circuital scheme; the forms of the derived equations of propagation of the second order will now be more complex, and must be deduced, in terms of the analysis appropriate to the varying manifold, from the fundamental scheme involving only first gradients. This illustrates in a special way the advantage of the familiar process of reducing a system of differential equations to a larger number of equations of the first order: the system thus becomes more amenable to considerations involving representation by extension in space. It is instructive to make an attempt to carry through the vector procedure.* If it fails when c' is variable it will mark the cause of failure. The available gradient operator, contragredient with R, is

$$\nabla_1 = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} + o\frac{\partial}{\partial \iota c' t};$$

but ∇_1^2 involving second gradients is not a scalar in the curved fourfold.

We start as before with a fourfold vector potential, now

$$U = iF + jG + kH + \frac{1}{\iota c'}\phi$$

determining the physical field. Then

$$\nabla_1 U = jk\alpha + ki\beta + ij\gamma + \frac{1}{\iota c}(io P + jo Q + ko R),$$

wherein

$$\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} - \frac{1}{c'^2}\frac{\partial \phi}{\partial t} = 0$$

$$\alpha = \frac{\partial H}{\partial y} - \frac{\partial G}{\partial z}, \quad \beta = \ldots, \quad \gamma = \ldots,$$

$$P = -\frac{c}{c'}\frac{\partial F}{\partial t} - \frac{\partial}{\partial x}\left(\frac{c}{c'}\phi\right), \quad Q = \ldots, \quad R = \ldots,$$

the latter equations introducing the magnetic field $(\alpha\beta\gamma)$, devoid of convergence, and the electric field (PQR) which converges upon the electric charges.

Again,

$$\nabla_1 \nabla_1 U = iA + jB + kC + o\frac{\iota}{c}D$$

$$+ jko A' + koi B' + oij C' + ijk D',$$

in which now

$$A \text{ represents } \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z} - \frac{1}{cc'}\frac{\partial Q}{\partial t},$$

* On the lines of "Generalised Relativity," *Proc. Roy. Soc.*, Nov. 1919.

$$\text{A}' \text{ represents } \frac{1}{\iota c}\left(\frac{\partial \text{R}}{\partial y}-\frac{\partial \text{Q}}{\partial z}\right)+\frac{\partial \gamma}{\partial \iota c' t},$$

$$\text{D} \quad ,, \quad \frac{\partial \text{P}}{\partial x}+\frac{\partial \text{Q}}{\partial y}+\frac{\partial \text{R}}{\partial z},$$

$$\text{D}' \quad ,, \quad \frac{\partial \alpha}{\partial x}+\frac{\partial \beta}{\partial y}+\frac{\partial \gamma}{\partial z}, \qquad \text{already null.}$$

We note that when c'/c is constant, A′, B′, C′, D′ vanish identically by virtue of their mode of deviation from the potential U: thus $\nabla_1 \nabla_1 \text{U}$ is then a simple fourfold vector, whose vanishing gives the remaining equations of the field in free space. In fact triple unitary products cannot arise, simply because $\nabla_1{}^2$ is a scalar; and likewise the square of any other vector occurring in a formula reduces to that simple form.

We have now to make

$$\frac{1}{4\pi}\nabla_1 \nabla_1 \text{U} = iu_1 + jv_1 + kw_1 + o\iota c'\rho,$$

involving that A′, B′, C′, D′ vanish, giving four of the equations of the field. Thus *if we may* operate a third time with ∇_1 in the element of extent, we obtain a scalar part

$$\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}+\frac{\partial w_1}{\partial z}+\frac{\partial \rho}{\partial t}$$

invariant for all change of the 4-hedral frame $ijko$. Hence, taking its value to be null, (u_1, v_1, w_1) can be a flux which accumulates a density ρ; that is, it can be expressed as $(\rho\dot{x}, \rho\dot{y}, \rho\dot{z})$.

The square of the vector thus limited to flux of electrons, namely of

$$\rho(i\dot{x}+j\dot{y}+k\dot{z}+o\iota c'),$$

is

$$\frac{\rho^2}{\delta t^2}(\delta x^2+\delta y^2+\delta z^2-c'^2\delta t^2),$$

of which the second factor is by itself invariant for changes of the frame $ijko$: hence so is $\rho/\delta t$; but $\delta x\delta y\delta z c'\delta t$ is invariant within the space-time element: therefore $c'\rho\delta x\delta y\delta z$, not the amount of charge $\rho\delta x\delta y\delta z$ but charge multiplied by c', is conserved as the charges move about, which is a contradiction or misfit (unless c' is constant) that seems unavoidable in this process. If this flagrant type of misfit is to be eliminated we must proceed by the exact but complex dynamical method of Action as *supra*.

We thus arrive at the modified electrodynamic scheme as follows:—

$$\frac{\partial \alpha}{\partial x}+\frac{\partial \beta}{\partial y}+\frac{\partial}{\partial z}=0, \quad \frac{\partial \text{P}}{\partial x}+\frac{\partial \text{Q}}{\partial y}+\frac{\partial \text{R}}{\partial z}=\frac{4\pi}{cc'}\rho.$$

$$\frac{\partial \text{R}}{\partial y}-\frac{\partial \text{Q}}{\partial z}=-\frac{c}{c'}\frac{\partial \alpha}{\partial t}, \quad \ldots, \quad \ldots,$$

$$\frac{\partial\gamma}{\partial y}-\frac{\partial\beta}{\partial z}=\frac{1}{cc'}\,\partial\mathrm{P}+4\pi\rho\dot{x},\quad \ldots,\quad \ldots,$$

Here $c'/c=1-\mathrm{V}/c^2$, through which the change in this electric scheme arising from the gravitational field V enters in the equations.

If it were legitimate to differentiate these latter circuital relations again, results would be obtained inconsistent with the first two equations of convergence. That is not surprising, because the curvature of the space would modify the values of second gradients. Neither can the slight defect of conservation of electric charge obtained above be an essential contradiction; it is ascribable to the same cause. And again, the vanishing of A′, B′, C′, D′ is inconsistent with the derivation of the vectors concerned in them from the potential U, unless c is constant; but here also the misfit arises from second gradients being dependent on the curvature.* Thus we conclude that the scheme of equations is valid, but not amenable to direct differentiation.

Even for the spherically symmetrical astronomical case, the Einstein special postulate, which shrinks the space towards the centre, will greatly complicate any such scheme.†

The General Formulation in Terms of Least Action.

The single variational equation expressing the interplay of relations of the general physical field is the analytical resource that is available for exploring our varying manifold, when more intuitive direct methods are thus at fault. The method is two-sided, including both the Riemann spacial scheme ‡ and the dynamics of the system that subsists in that space. Its form for the present electrodynamic purpose is

$$\delta\int \mathrm{L}d\tau=0,$$

* It is, as I understand it, this necessarily intricate but direct procedure, by mobile "geodesic differentiation" of the vectors, that is developed in an important memoir (95 pp.) by Professor J. A. Schouten, of Delft, in *Amsterdam Transactions*, xii., Feb. 1819, for a copy of which I am indebted to the author. The memoir has at the end a list of the very copious recent Continental literature.

† The mode of analysis here inadequately approached for a very special case has, I find, been effected in its generality in symmetrical and most beautiful manner by Th. de Donder in a memoir written at various times at Brussels in 1914–16 (see a note in *Comptes Rendus*, July 6, 1914), and published by H. A. Lorentz in *Archives du Musée Teyler* (Haarlem), vol. iii., 1917, pp. 80–179. The symbolic products of $ijk\ldots$ of the various orders in the algorithm above employed pass over into a realised geometrical interpretation as unitary components of integral invariants. From the binary integral form corresponding to $\nabla_1\mathrm{U}$ the principle of duality derives a conjugate one with regard to the binary form representing $\delta\sigma^2$: and the equations of the field find invariant expression in terms of them. From this scheme of analysis and the Least Action principle a set of field equations of stress-tensor, momentum, energy was readily developed.

‡ The variation of the Riemann parameters, introduced by Hilbert, is the mode of procedure.

in which the variation is to be carried through subject to the structural relations of the containing fourfold. Here $\delta\tau$ is the element of extension of type $\delta x\delta y\delta z c'\delta t$ in the curved fourfold, these factors being analogous to Gaussian elements of length on a curved surface; and $-\mathrm{L}$ the kinetic potential per unit volume in ordinary space is given by

$$\mathrm{L} = \mathrm{H}^2 - c^{-2}\mathrm{F}^2$$

in which H is magnetic and F electric force, this Lagrangian function (of density L) being invariant for internal transformation of co-ordinates in each fourfold element $\delta\tau$, and so being a physical entity independent of co-ordinates; while $\delta\tau$ is also thus invariant.

Now, the necessity for recognising a force of gravitation is supposed to arise from an illusion by which the co-ordinate elements δx, δy, δz, $\delta c't$ which thus really belong to the fourfold element in non-uniform curved extension, envisaged as existing in the flat fivefold, are tacitly presumed to belong to an element in a flat fourfold, with resulting misfit of these elements of extension as if one tried to build on the flat with bricks made for a curved arch. It is thus indicated that we might, perhaps we ought to, complete the illusion by transferring this all-including electrodynamic equation of Action, just as it stands, to the flat in the same way. This will involve that the values of H and F are simply transferred without change to the corresponding points on the flat, *and that the value of* $\delta\tau$ *is so transferred without change.*

In this way of proceeding, postulated for the sake of simplicity alone, the correspondence between the presumed actual curved fourfold and its forced representation in the flat, including the necessary accompanying illusion of natural constraining forces of gravitation, is to be such that its invariant element of fourfold extension $\delta\tau$ is to retain the same value in the flat representation. As has been concluded above, it is this extraneous demand on the scheme of closest representation that introduces the slight discrepancy from Newtonian gravitation which need not otherwise have been involved.

Conclusion.

We may now sum up the point of view into which we have been driven provisionally as a direct and possibly not entirely inadequate interpretation of these analytical formulations, so far as they are of immediate practical scope. The world-process is to be regarded as made up of physical vectors called the electric force and the magnetic force, and the electric nuclei which are the poles of the former, and material particles which ought to coincide with these complex electric nuclei and represent another aspect of them. This very limited cosmos, so specified, is located in a curved fourfold frame of space and time, which is suitably envisaged as a section of an auxiliary flat fivefold involving free mobility, and therefore permitting transferable auxiliary measuring

systems such as we require in some form or other in order to establish ideas of fourfold comparison and location. When we forbear from locating in this flat auxiliary fivefold, we can hardly avoid, in framing theories, the temptation to simplification by a wrong representation for the world, by locating it instead in an imagined flat fourfold frame with its simple possibilities for measurement, in which the system of nature will really not fit; and the specification of the character of the misfit then appears through additional terms arising in the variational dynamical process, that are interpretable as new compelling universal forces such as gravitation; they may even be the whole of such forces, which would in this manner be cleared away altogether from the scheme of nature at the cost of an equivalent complication of the frame of space and time. The result, as regards all particles whose course can be traced, will be that though actually travelling free in shortest lines in the curved section of flat fivefold in which the universe subsists, they will be interpreted as describing gravitational orbits in its wrongly assumed flat fourfold location. As regards the presumably ultimate electrodynamic vectors, they could hardly carry over from the curved into the imagined flat fourfold without change of their essential interlocked constitution beyond recognition, and probably without extraneous complication, unless the element of extension in the hypothetical flat fourfold can be restricted to its previous value in the curved fourfold from which it is carried over. Then the fundamental electrodynamic and optical constitutive relations in the flat fourfold illusion will be reasonably similar to but not identical with the standard simple form they would have retained if there had been originally a true flat fourfold, with its ideally simple schemes of measurement, instead of the actual curved one.

The implication, in any such correspondence as this, is that the electric force and the magnetic force are regarded as real ultimate physical entities as well as the permanent electric ions which they determine and locate; and that their distribution is carried over without change * along with the latter from the curved fourfold extension to which phenomena actually belong into an imagined flat fourfold in which we insist on locating them. This transfer is perhaps permissible even as regards such abstractions; though it may be regarded as very artificial in the earlier light that identified matter and electrons as the sole permanent entities, in terms of which all the relations which constitute physical knowledge are ultimately determined.

Thus gravitation can be absorbed quite naturally and simply into a frame of varying space and time without involving alteration of the Newtonian law to any degree that could be detected. But if the theory of the electrodynamic field with its two fundamental intertwined vectors, electric force and magnetic force, and the related electric poles, and its relativity with regard to mixed space and time, and the whole doctrine of physical optics with its

* The Least Action procedure avoids such assumption of mere transfer.

essential feature of propagation by transverse vectors, are to pass over also in any recognisable manner, the varying curved continuum of space and time will have to be restricted so that its element of extension shall be equal to the corresponding element in the flat of the usual presumably imperfect formulation. It is this latter condition, introduced for the sake of electrodynamic simplicity alone,* that complicates the law of Newtonian gravitation, and at the same time doubles the deflection of a ray of light that is a consequence of the spacial absorption of that force.

The theory as resumed by Einstein in his summary of 1916 (*Ann. der Physik*, 49), the latest that is accessible, does not seem to contemplate interlinking of detailed electrodynamics and optics, at any rate at that stage, with the modified gravitation, through a conjoint variational formulation of the dynamical principles or otherwise. When the gravitational scheme has been developed into relation with the earlier principles of mechanical relativity which, for speeds almost outside our experience, transcends the mass of a body by identifying it with an aspect of its energy, and therefore when in that scheme matter is everywhere eliminated, and replaced by flux of momentum and energy and the stresses which modify them, the question arises, how is the electrodynamic field to enter into it also? The answer (§ 20), at any rate provisional, appears to be naïvely as follows: let there be a fourfold potential, form by algebraic process a six-vector from it, being its gradient in Clifford's symbolic fourfold calculus with respect to the four positional variables—but these variables are not spacial co-ordinates and so *are not components of a vector*, which seems to abstract all meaning from that algebraic process in any physical invariant sense. But whether this procedure be a link in the evolution of a theory or a mere provisional stop-gap, the ultimate result as sought and attained is the inclusion of electric energy and momentum along with the apparent gravitational quantities of like types in the analytical equations that now express gravitation. The analysis thus seems to aim primarily at being a theory of the flux of energy and momentum, which does not comprehend or explain the physical electrodynamic field but is a feature of it. The crucial postulate on which all hangs, the conservation of the element of fourfold extension in the transformation of one space-time scheme to the other, appears to be left without any support except this plea of simplification. It is on this analytical postulate that the new verified effects depend.

The mode of interpretation we have been driven to assign to Einstein's analysis thus rests on the presumption that in the inclusion of electrodynamics the circulatory interlocked relations of the physical field of Maxwell are too primary and fundamental, and present to the mind too crisp and simple an idea of mutual structure, to be tampered with by any arbitrary modification of space and time; in fact, to amend them with due respect and

* It may also be held to be necessary to assure validity of the approximations which lead for the actual world to Newtonian gravity.

coherence it is necessary to go back to the variational presentation of universal dynamical science in which they originate.

Whatever view one may take, these analytical theories demand close attention and scrutiny from all sides, as they have supplied the first probable clue to the nature of gravitation. But still more arresting even than they have proved to be, for general views, is perhaps the fact that the original theories of space-time correspondence, which were constructed for the correlation of identical systems moving with different planetary speeds, small compared with the velocity of light, are also confirmed as regards one feature by no means small, the inertia effect derived from the correlation, for the structure constituting the unique and unvarying electron whatever that may be, when it is travelling with speed even as much as four-fifths of that velocity.

It may help to exhibit a connected view if we try to summarise a conceivable order of evolution of ideas, in the very inadequate medium of ordinary language, through which however new ideas originate, to be developed and tested later by more complex means. The analytical frame of mixed space and time formulated by Minkowski is admirably adapted to concise expression of the invariance or relativity inherent in electrodynamic theory. For the case of a radial symmetrical field of gravitation it can further absorb that universal force merely by making the scale of time vary with position. The *additional* postulate of conservation of infinitesimal fourfold extents then requires a radial shrinkage of the space, which in the absence of an equal transverse shrinkage destroys its flatness: it is this that modifies gravitation. If we wish to avoid this deformation of our flat space,* which is only partially thinkable, it is open to us to describe the same as change of radial inertia with speed of the moving bodies, in extension of previous electrodynamic ideas regarding inertia; but then the doubled deviation of light becomes more obscure. The form of Least Action that has been used permits the potential of the field to involve the time explicitly; but the sources of the gravitation are to be practically stationary in comparison with the velocity of radiation. These limitations stand in the way of partial generalisations. The course that is open is to start afresh with the widest possible foundation on the same lines, to eliminate inertia altogether, as it otherwise becomes mixed up as above with the varying scales of time and space, and to hold fast to the equations that remain—those of conservation in the transfer of energy and momentum in the field, subject to the activity of a working stress in the medium of the type of the previously rather accidental Maxwellian æther-stress which thus achieves a fundamental significance. This general theory, of immense analytical complexity, was apparently not required to predict the new effects on orbits and on rays, but it is the extension into conditions beyond present

* We could change the generalised co-ordinates to new ones, making $\delta\sigma^2$ isotropic: but it may be held that we must abide by the set which makes closest correspondence with actual astronomy.

experience of a very remarkable and novel train of ideas into which these effects can be made to fit.

Appendix.—*On Representation in Space.*

An explicit algebraic transformation such as $x_1 = f(x, y, z, \ldots)$ changing

$$\delta\sigma^2 = \delta x^2 + \delta y^2 + \delta z^2 + \ldots$$

into a quadratic expression for $\delta\sigma^2$ of wider form does not alter the curvatures of the spacial manifold determined by $\delta\sigma^2$; being originally flat, it remains flat for the new form. But a correspondence between forms of $\delta\sigma^2$ whose connection is not expressible as here by explicit algebraic equations involves essential changes in the manifold. Algebraic connection between the two could then be established only by our including them both in the same manifold of higher dimensions. The nature of the space depends on the formula for $\delta\sigma^2$; but it persists for all types of formula that are derived by algebraic transformation of the co-ordinates. The new co-ordinates represent the same point in the space as the original ones. The significance of these statements may be paraphrased as follows. Knowledge consists of the play of relations; but to have relations, the permanent entities must be assumed, in a background outside actual perception, which are related. Relations are expressed by equations within the set of co-ordinate quantities: when these co-ordinates are changed to any new set whatever, derived from the former in some assigned manner, the relations persist, remain invariant, though the equations expressing them in terms of the co-ordinates have become quite different. But now the restriction comes in, that those co-ordinates are not entirely arbitrary but refer the system to a measured spacial manifold, in which it is, in the usual figurative language, located. The theories here under discussion take it to be a manifold that is measurable in the Riemann way; there is an element $\delta\sigma$ of *absolute* length (or less ambiguously it may be called absolute time, as that has only one dimension; or it may be called by any other name, for it is quite extraneous) which is expressed everywhere by a quadratic function of differentials of co-ordinates, but which is transferable in the manifold without change of value. The Riemann differential theory proceeds by transferences of $\delta\sigma$ *restricted* to adjacent positions, and finding the relations which the various gradients of its co-efficients must satisfy in order that its value may remain unchanged. Thereby the conditions for a uniform manifold can be brought out, that is, for one which is of the same kind around all points in it, such as a plane or a sphere; if it is not uniform the essential or invariant quantities are determined which specify its quality around each point in a region small enough to be taken as uniform—that is, which determines its intrinsic "curvature," thus expressing its

essential structure.* Now it is certain that it is only relations, and the permanent entities that are related, that have absolute significance: that the set of co-ordinate quantities in terms of which the relations are expressed are accidental, can be changed into a new set by any scheme of correspondence that we please, even one which introduces, by essential change in the form of $\delta\sigma^2$, a change of the nature of the space of reference. The choice even of space might thus be arbitrary; it is only the relations in that space that are of real account.

If this be so, need a change of the kind of space modify the law of gravitation at all? The answer would be that there is one change of space that is proper to each change of form of $\delta\sigma^2$, and will not alter anything; but that if we are tempted from motives of simplification to assert a change differing from this proper one, the relations which represent the laws of nature will be deranged, so that the laws must be thereby modified.

A transformation has been specified above which absorbs gravitation into space practically without modifying it at all. It must be, then, that this proper transformation leads to more drastic change in other respects in the expression of natural relations than we find consistent or convenient; actually and ultimately that it prevents the laws of physics from being expressible, in the form which ensures consistency automatically, as one all-embracing variational equation after the type of Least Action.

In order to obtain correspondence simple enough to be analytically manageable, it might be presumed that an element of the transformed space must remain adjacent to its correlative element in the original space, when both are located in the same containing flat fivefold, in the sense that their two sets of co-ordinates in the fivefold differ only by small quantities. This may be essential to a workable approximation, such as the one that represents the Newtonian gravitation under the limited conditions of actuality. This condition of proximity could not be satisfied unless corresponding elements of fourfold extent are equal, up to the first order of small quantities: though this equality alone will not suffice. In the polar expression for $\delta\sigma^2$ for radial fields, as above, it is not satisfied until δr^2 is affected by a factor c^2/c'^2. It is this secondary factor that complicates the correspondence with the Newtonian law of gravitation. But it is no absolute necessity of logical deduction that has led to this result: it rather seems to be part of the first approximation that assumes small changes in the straining of the space in the fivefold—in disregard of the Laplacian *dictum* (probably misleading in any profound sense) that nature takes no heed of the difficulties of analysis.

To resume. If the laws of nature are defined as the relations that are invariant in one scheme of space and time, the same

* In a fourfold there are six components of local curvature, but only one invariant quantity that seems (Hilbert) of sufficient simplicity to form the basis of a practicable variational theory, and thence of a stress-tensor scheme after the Einstein model.

relations cannot be other than invariant when nature is transformed correctly into any other scheme: the laws cannot alter by change of their frame. But the correspondence is broken if the equations that belong to one frame are treated, unchanged, as if they belonged to a different one. The absorption of gravitation into spacial relations can be effected exactly, if so desired. But corresponding elements will not be all adjacent in the fivefold if the element of fourfold extension does not remain unchanged to the first order; and without this proximity no approximation to the practical theory of gravitation might be safe. And other relations of nature also, whose formal type is known, would be too greatly disturbed. The development, thus happily restricted to practicable bounds, would then centre around an extended Maxwellian guiding scheme of stress-energy-momentum in a medium, or more fundamentally and completely, the postulate of a single variational foundation for physical phenomena after the manner of Least Action.

On the Crucial Test of Einstein's Theory of Gravitation.

By Harold Jeffreys, M.A., D.Sc.

I. The Relevant Data.

1. *The Motions of the Inner Planets.*

On the basis of his theory of gravitation Einstein made three predictions of phenomena capable of observation, namely: (1) the perihelion of Mercury should advance by 43″ per century, (2) a ray of light just grazing the sun should be deflected by 1″·75 towards the centre of the sun, (3) dark lines in the solar spectrum should be displaced to the red by an amount corresponding to a velocity of 0·634 km. per second. It is claimed that the first two of these have been verified; but before the numerical agreements found are accepted as confirmations of the theory, it is necessary to consider whether there are any other causes that could produce effects of the same character and greater in magnitude than the admissible error. Suppose, for instance, that a true cause was known that would produce a motion of 10″ per century in the perihelion of Mercury; then Einstein's theory would predict the total excess motion to be 53″ per century, which differs from the observed 41″ by more than the permissible error of observation. Such a discovery would be fatal to a theory such as Einstein's, which contains no arbitrary constituent capable of adjustment to suit empirical facts. Now a sufficient amount of gaseous matter within the orbit of Mercury would be capable of producing both the first two of Einstein's effects—the first by its gravitation, and the second by its refraction; and such matter is known to exist, causing the solar corona and the zodiacal light.

It is therefore desirable to inquire whether its quantity is sufficient to invalidate the theory. The question appears to be capable of a direct answer in the negative, for the reasons that will now be given.

Let C be the greatest and A the least moment of inertia of such a mass about axes through the centre of the sun. Take the mass of the sun as the unit of mass and the earth's mean distance from the sun as the unit of length. If $\beta = 0{\cdot}648 \times 10^8(C - A)$, I have shown* that the motion of the perihelion of Mercury can be completely accounted for if β is as much as 0·575. With suitable assumptions about the position of the axis of greatest moment of inertia it is also possible to account for the other unexplained secular perturbations of the inner planets; thus such a distribution would have strong claims to acceptance. It shows at once that if the motion of the perihelion of Mercury is to be explained by means of gravitating matter not yet allowed for, we must have

$$C - A = 0{\cdot}886 \times 10^{-8} \quad . \quad . \quad . \quad (1)$$

Let us consider now whether the zodiacal light gives us any reason to infer such a value of C – A. Let ω denote the solid angle subtended by a circle of apparent diameter 1°. The ratio of the amount of light in such a circle of the zodiacal light to the intensity of sunlight is $3{\cdot}7 \times 10^{-13}$. On the other hand, the ratio of the skylight in an ω at Mount Wilson to the intensity of sunlight is 3×10^{-6}. Thus the intensity of the zodiacal light is $1{\cdot}2 \times 10^{-7}$ of that of skylight at Mount Wilson. Now the intensities of the light received by the zodiacal matter and the air are in a ratio of the order of 2 to 1, and if they have equal scattering power, which seems unlikely to be far out, the masses in columns of equal cross section would therefore be in the ratio of $0{\cdot}6 \times 10^{-7}$ to 1. From the atmospheric pressure at Mount Wilson and the thickness of the zodiacal matter we at once infer that the density of the latter is of order $1{\cdot}5 \times 10^{-18}$ gm./cm.^{3}. Then C – A is of order $1{\cdot}5 \times 10^{-18} r^5$, where r is the radius of the zodiacal matter. In the above units this is equal to $2{\cdot}5 \times 10^{-12}$ even if the matter extends out as far as the earth's orbit. This is only $\frac{1}{3000}$ of the amount required by (1) to give the motion of the perihelion of Mercury, and could produce no observable gravitational effect on it or any other planet.

The above treatment assumes the zodiacal light to be scattered by the molecules of a gas. It was formerly believed to be diffusely reflected from solid particles. In this case the luminosity is, however, shown by the method of my former paper to correspond to a value of $\nu\sigma^2$ of $3{\cdot}5 \times 10^{-22} \div 1$ cm., where ν is the number of particles per unit volume and σ their average diameter. C – A is of order $\rho\nu\sigma^3 r^5$, where ρ is the density of a particle. Now σ cannot be greater than 10 km., for the particles would then be visible as separate inner planets. Hence C – A is not greater than

* *Monthly Notices of R.A.S.*, 77, 1916, p. 117.

5×10^{-10}, and this could not produce a perihelion motion of Mercury greater than 2″ per century, which could not be certainly observed. Thus this hypothesis, even if correct, could afford no argument against Einstein's theory; while there are cosmogonical arguments against it in any case, on account of the comparative shortness of the time such a distribution of solid bodies could persist in spite of collisions.

Near the sun the density of the matter present is shown to be much greater by the solar corona. Most of the light from this is scattered sunlight, while that from its outer portions seems to be entirely so. It shows a marked irregularity of form, for it usually extends furthest from the sun around the equator, not so far from the poles, and in intermediate latitudes least of all. It therefore seems at least possible that it could give a notable value of C − A. Further, on account of the rapidity with which its luminosity increases towards the sun, it appears to be just such a mass as could cause by its refraction a displacement of star images of the sort observed on May 29. Its apparent luminosity gives a means of testing these suggestions.

Let r denote the distance of a point in the corona from the centre of the sun, and θ the angle between the line joining this point to the centre and the line from the earth to the sun. Consider an element of volume $r^2 \sin\theta\, dr\, d\theta\, d\phi$, ϕ being the azimuthal co-ordinate. I being the intensity of radiation at the surface of the sun, this element will scatter in the direction of the earth an intensity $\frac{I}{r^2}\kappa\rho(1+\cos^2\theta)r^2 \sin\theta\, dr\, d\theta\, d\phi$, where κ is a constant depending on the nature of the matter present and ρ is the density. Now, consider all those molecules for which $r\sin\theta$ lies between ϖ and $\varpi + d\varpi$, and ϕ lies within the fixed range $d\phi$. They will give the intensity of radiation from a fixed apparent position relative to the sun. This, absorption being neglected, shows that the radiation received from unit apparent area for a given r and ϕ is

$$\int_{-\infty}^{\infty} I\kappa\rho(1+\cos^2\theta)r^{-2}dx \quad . \quad . \quad . \quad (2)$$

where

$$x = r\cos\theta = \varpi\cot\theta \; . \quad . \quad . \quad . \quad (3)$$

Substituting for x and r in terms of ϖ and θ we have

$$2\int_0^{\frac{1}{2}\pi} I\kappa\rho(1+\cos^2\theta)\varpi^{-1}d\theta \quad . \quad . \quad . \quad (4)$$

Now, suppose as a first approximation that ρ is proportional to r^{-n}, where n is a positive integer. We have then

$$\rho = \alpha r^{-n} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$= \alpha\varpi^{-n}\operatorname{cosec}^{-n}\theta \; . \quad . \quad . \quad . \quad (6)$$

The apparent brightness is then

$$\kappa a \mathrm{I} \int_0^{\pi} \varpi^{-n-1} \sin^n \theta (1+\cos^2\theta) d\theta = 2\kappa a \mathrm{I} \varpi^{-n-1} \lambda \frac{n+3 \cdot n-1 \cdot n-3 \cdot n-5 \cdots}{n+2 \cdot n \cdot n-2 \cdot n-4 \cdots} \quad (7)$$

where λ is to be put equal to 1 if n is odd, and to $\frac{1}{2}\pi$ if n is even.

Professor Turner has stated * that the luminosity of the corona in the 1898 eclipse could be represented by the formula

$$A\varpi^{-6} + B \quad . \quad . \quad . \quad . \quad (8)$$

where A = 2·2 times the luminosity of the full moon,
B = 0·012 times the luminosity of the full moon,

and the unit of length is the radius of the sun. He considers that only the first of these terms is really due to the corona itself, the constant term being caused by the general luminosity of the sky and the zodiacal light. Thus for our present purpose we need consider only the first; we see that the conditions are all satisfied if $n = 5$ and

$$\tfrac{128}{105}\kappa a \mathrm{I} = 2{\cdot}2 \text{ (full moon)} \quad . \quad . \quad . \quad (9)$$

The ratio of the intensity of skylight at Mount Wilson to the intensity of the light from an equal apparent area of the sun is $0{\cdot}7 \times 10^{-7}$; taking the luminosity of the moon to be 1/150,000 of that of the sun, we see that $\kappa a \mathrm{I}$ must be equal to 200 times the luminosity of the sky. The luminosity of the sky again is of order $\mathrm{IR}^{-2}\kappa\rho\mathrm{H}$, where R is the distance of the earth from the sun, κ and ρ are evaluated for air, and H is the effective height of the atmosphere. We can legitimately take κ to be of the same order for the corona and air; R and H must of course be evaluated in solar radii. We get then

$$\rho = 0{\cdot}32 \times 10^{-11} r^{-5} \frac{\mathrm{gm}}{\mathrm{cm}^3} \quad . \quad . \quad . \quad (10)$$

where the unit of length is the solar radius.

Now, for the corona

$$\mathrm{C} - \mathrm{A} = \tfrac{8}{3}\pi \int e\rho r^4 dr \quad . \quad . \quad . \quad . \quad (11)$$

where e is the ellipticity of the stratum of density ρ. In the corona it seems to be of order $\frac{1}{5}$. Thus

$$\mathrm{C} - \mathrm{A} \text{ is of order } 5 \times 10^{-12} \frac{\mathrm{gm}}{\mathrm{cm}^3} (\text{sun's radius})^5 \quad . \quad (12)$$

Even if the corona is considered to extend as far out as 20 solar radii this amount will only be trebled. In the units we used in (1) this becomes 10^{-17}, an insignificant fraction of the amount shown to be needed to produce any important effect on the motions of the inner planets.

Thus neither the zodiacal matter nor the corona is quantitatively

* *Monthly Notices*, 61, Appendix, pp. 4–12 (1900).

competent to affect Mercury and Venus sufficiently to furnish any argument against Einstein's theory. The excess motion of the node of Venus, if confirmed by future observations, cannot then be accounted for by the attraction of intra-Mercurial matter; at the same time, now that such matter is known not to exist in sufficient quantities to affect these planets, the fact that the excess motion of the node of Venus requires a considerable quantity to account for it affords an argument against the reality of this excess motion, but none against the theory of Einstein; for this and the Newtonian theory alike are now shown to require that such a motion should not exist.

2. *The Displacement of Star Images.*

The zodiacal matter is too scattered to produce any appreciable refraction on light passing through it, but the rapid increase of density in the corona towards the sun suggests that refraction there may be noticeable. If n denote the refractive index of the gas, and ϖ be the shortest distance from the ray to the centre of the sun, the inward deflection (corresponding of course to an apparent displacement of a star away from the sun) is

$$-2\varpi\int_{\varpi}^{\infty}(r^2-\varpi^2)^{-\frac{1}{2}}\frac{dn}{dr}dr \quad . \quad . \quad . \quad (1)$$

Now, in this case $n-1$ is probably proportional to ρ. For air $(n-1)/\rho=0{\cdot}22$; for hydrogen it is $1{\cdot}52$. Using the value of ρ already found, we find for a corona of hydrogen a deflection

$$10\varpi\int_{\varpi}^{\infty}(r^2-\varpi^2)^{-\frac{1}{2}}1{\cdot}52\times 0{\cdot}32\times 10^{-11}r^{-6}dr=0{\cdot}8\times 10^{-11}\varpi^{-5} \quad (2)$$

Thus the deflection at the limb, where by convention ϖ is 1, is $1''{\cdot}6\times 10^{-6}$, a millionth of that observed. The corona probably does consist mostly of hydrogen; but for most gases $(n-1)/\rho$ is less than for hydrogen, so that if this assumption is incorrect we shall have overestimated the deflection. Further, the actual deflections were measured at distances greater than twice the sun's radius; and while Einstein's deflection is proportional to ϖ^{-1}, the refractive deflection is proportional to ϖ^{-5}; thus if the observed displacements were due to refraction, the displacement at the limb would be over sixteen times Einstein's predicted amount (*i.e.* 28″), in still more serious disagreement with the amount possible. Refraction is, therefore, totally incapable of accounting for the observed displacements.

3. *The Displacement of Spectral Lines.*

St. John and Evershed have sought for verification of the shift of spectral lines predicted by Einstein, without success. Whether this fact can be regarded as a serious objection to Einstein's theory is, however, very doubtful. The outstanding

features of the differences observed by Evershed between the wave-lengths of lines in the spectrum of the limb of the sun and in that of the arc are, first, an almost complete uniformity of direction in the shifts; and second, a striking variability in amount. Thus in *Kodaikanal Bulletin*, No. 39, Evershed gives a list of shifts of iron lines ranging from −0·003 A to +0·037 A, while Einstein's prediction is about 0·008 A, varying somewhat according to the region of the spectrum. These shifts do not vary much according to the intensity of the lines, and it is shown that the displacements to the red are on the whole greatest for the lines least affected by pressure. This in itself constitutes evidence that if the shifts are to any considerable extent influenced by pressure, the pressure in the reversing layer is less than one atmosphere, and that the shifts corrected for the pressure effect would be still more strikingly positive. The mean for all the lines is about 0·013 A. For cyanogen, again, Evershed finds (*loc. cit.*, p. 79) a mean shift of 0·008 A, precisely the Einstein amount. He accounts for these phenomena by supposing a general motion in the visible part of the sun in the direction away from the earth; but there seem to be difficulties in the way of accepting this hypothesis apart from the implied attribution to the earth of special properties which are not shared by the other planets. The downward motion suggested to account for the displacements at the centre of the sun cannot be constant over the solar surface, for it would mean that the whole of the reversing layer would be reabsorbed into the sun in a few hours. There must be an upward motion on the side remote from the earth to balance it. Hence when Venus is on the opposite side these gases must be approaching it, and accordingly the lines in the spectrum of light reflected from it must be displaced to the violet. Evershed has observed Venus in such a position (though not for this purpose) and finds no evidence of a shift. If, however, the Einstein effect exists, the excess displacement above it may perhaps be accounted for in Evershed's way, and the two effects will at least partially neutralise each other for Venus. Thus with a slight modification of Evershed's theory, his results may be held to support Einstein.

There is, however, a further difficulty in the way of the theory that a large part of the shifts is due to Doppler effects. It is suggested by St. John (and Evershed seems prepared to entertain the suggestion) that the intensity of lines are closely correlated with the depth at which they are produced. The velocity of the gases at the limb is presumably a function of the depth, and the displacement should therefore be reasonably constant for lines of the same intensity, contrary to what was found. The range of variation may be judged from the following table of the limb−arc displacements of iron lines of intensity 5 between λ = 4871 A and λ = 5616 A.

λ.	Displacement.	λ.	Displacement.	λ.	Displacement.
4871·512	0·013	5167·678	0·033	5281·971	−0·002
4903·502	0·017	5192·523	0·002	5302·480	0·000
4924·107	0·014	5227·362	0·006	5434·740	0·018

Evershed's observations of the cyanogen lines are, however, in violent contradiction with those of St. John,* who finds an average displacement of these lines of +0·0018 A, which apparently contradicts Einstein's theory. The question is not, however, a mere discrepancy between observation and theory, for St. John and Evershed have here observed precisely the same phenomenon and obtained results differing by about ten times the probable error.

St. John and Babcock † have also observed the displacements of iron lines, and shown that when the pole effect in the arc is removed, the differences between the displacements of different lines are very much reduced. For two groups of lines, average displacements of +0·007 A and +0·008 A are found, agreeing perfectly with Einstein, but showing that under the laboratory conditions at Mount Wilson the observed shifts of iron lines differ considerably from those of cyanogen lines.

It appears to me, therefore, that it is impossible to assert definitely that the predicted shift of spectral lines is or is not confirmed by observation. The discordances between observations in different conditions are so great as to require explanation before any decision on the theory is possible. One theoretical point may nevertheless be mentioned. Einstein's prediction rests essentially on the assumption that the radiating atom is a natural clock whose period is the same on both the sun and the earth. While this is probably true of an isolated atom, the actual atom is very much influenced by its surroundings, and these may affect the periods. In particular, if any coupling of vibrations occurs, some periods will be increased and others diminished, and the variation in the shifts will be accounted for. The existence of such physical influence on the periods is shown in purely laboratory conditions by the existence of pressure shifts and the pole effect in the arc; and in the sun, where the theory of stellar evolution demands the transmutation of elements on a colossal scale, it would be very surprising if the periods of atomic vibrations were not affected during the intervals of stability. We should then get non-gravitational influence on the shifts in both directions, probably as often positive as negative, while the Einstein effect would show in the average of all, as it actually appears to do in several of the sets of observations.

II. The Relation of Einstein's Theory to Observation.

The verification of two of the crucial tests of the theory renders it possible now to attach a considerably higher probability to the theory than was possible from the original presentation; for Einstein's formulation is essentially dependent upon an assumption that can in no sense be regarded as obvious. The empirical results that have given new information about the nature of

* *Astrophysical Journal*, 46, November 1917.
† *Ibid.*, 46, 161 (1917).

gravitation can be used to remove this assumption, which now becomes one of the interesting mathematical consequences of the theory.

Einstein's 1905 theory of relativity can be made to rest on two fundamental assumptions concerning observations made relative to two non-rotating and physically isolated sets of axes.

(1) Any isolated particle moves with uniform velocity in a straight line in either system.

(2) The measured velocity of light is the same in either system.

From these data, which can be regarded as well established and in accordance with observation, it can be proved deductively * that if x, y, z be the position co-ordinates of a point where an event happens in the first system at time t, and if the same event happens at the point x', y', z' relative to the second system at time t', the accented quantities can all be found as linear functions of the unaccented ones, with determinable coefficients, and satisfy the differential relation

$$c^2dt^2 - dx^2 - dy^2 - dz^2 = c^2dt'^2 - dx'^2 - dy'^2 - dz'^2 \quad . \quad (1)$$

where c is the velocity of light. This may be written

$$ds^2 = ds'^2 . \quad . \quad . \quad . \quad . \quad (2)$$

Thus ds is the same whatever axis we refer our observations to. Further, the motion of a particle is given by the condition that $\int ds$ is not changed by small variations in the (four-dimensional) path, the ends being kept fixed. Thus the conditions of motion of a particle can always be put into this form whatever the system of reference may be.

Now in a gravitational field particles cease to move in straight lines, so that the relation

$$\delta \int ds = 0 . \quad . \quad . \quad . \quad . \quad (3)$$

ceases to hold for the motion of a particle so long as ds has the above value. Let us, then, suppose that the expression whose variation is now zero is

$$K \int ds \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where K is some function of x, y, z, and t, and the new ds is defined by the equation

$$ds^2 = \sum_i \sum_j g_{ij}\, dx_i\, dx_j \quad . \quad . \quad . \quad (5)$$

where the g_{ij}'s are functions of x_1, x_2, x_3, and x_4, which are themselves functions of x, y, z, and t. This is so far rather more general than Einstein's corresponding assumption.

Now, we can easily show that K is irrelevant. For if the

* *Phil. Mag.*, 38, 349–351, 1919.

suffix zero denote initial conditions and suffix unity denote final conditions, we find that for a small variation in the path

$$\delta[\mathrm{K}\int ds] = \mathrm{K}_1\delta\int ds + [\delta\mathrm{K}]_1\int_0^1 ds.$$

If by hypothesis the final position and time are given, $[\delta\mathrm{K}]_1$ is zero, and therefore unless K_1 is zero, which cannot be true in general, $\delta\int ds$ will be zero for any small variation in the path. Thus Einstein's assumption that the equations of motion are derivable from

$$\delta\int ds = 0 \quad . \quad . \quad . \quad . \quad . \quad (6)$$

is deducible from any of the apparently more general forms just suggested.

It may now be shown that the general co-variance of the equations of motion of a particle involves no further assumption. For suppose we have another system of reference, then with regard to this system the motion of a particle is given by a condition of the form

$$\delta\int ds' = 0 \quad . \quad . \quad . \quad . \quad (7)$$

where

$$ds'^2 = \sum_i \sum_j g_{ij}' \, dx_i' \, dx_j' \quad . \quad . \quad . \quad (8)$$

The g_{ij}''s are so far unknown. Now the motion of the particle is the same whatever may be the system of reference. Hence the conditions for the vanishing of $\delta\int ds$ also imply the vanishing of $\delta\int ds'$. There is so far, however, no reason to believe that ds and ds' are actually equal. We have, however, evidently $ds' = \lambda ds$, where λ is some function of x_1, x_2, x_3, and x_4.

Now suppose, if possible, that λ is not a constant. Then it will be possible to choose a path so that $\int ds$ is stationary for small variations in it, and such that λ can be altered by such small variations. Then if we choose a varied path such that it coincides with the original path where λ would be unaltered or diminished by a variation, and such that a variation increasing λ takes place where such increase is possible, $\int \lambda ds$ and therefore $\int ds'$ will be increased by such a variation. Similarly unless λ is a constant we can always show that the vanishing of $\delta\int ds$ would not imply that of $\delta\int ds'$. Hence λ must be a constant, and without loss of generality, by a mere change of units, we can make it unity. Thus, whatever be the system of reference,

$$ds' = ds^* \quad . \quad . \quad . \quad . \quad . \quad (9)$$

* No assumption is made concerning the identity of ds with the line element in the 4-dimensional space-time, or connecting the g_{ij}'s with the "curvature" of such space-time. Space, time, and curvature are metaphysical conceptions whose utility, if any, depends wholly on the individual peculiarities of the person trying to understand them. The quantities here considered are those derived from physical measurements, so that the invariance of ds is no longer axiomatic.

The equations of motion are, therefore, expressed in the condition

$$\delta\int ds = 0 \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where ds represents the same quantity in all systems. This equation can by the methods of the calculus of variations be made to yield at once four differential equations of the form

$$\frac{d}{ds}\left(\sum_j g_{ij}\frac{dx_j}{ds}\right) - \sum_j\sum_k \frac{\partial g_{jk}}{\partial x_i}\frac{dx_j}{ds}\frac{dx_k}{ds} = 0 \quad . \quad . \quad (11)$$

These are evidently obtained from the co-ordinates and the g's in precisely the same way, no matter what may be the system of reference. In other words, the equations of motion are generally covariant, and one of Einstein's fundamental assumptions is proved.

On the other hand, this tells us precisely nothing about the actual form of the law of gravitation. Whatever the g_{ij}'s may be, ds can always be expressed in the form (5), and therefore if ds^2 is any quadratic differential form whatever it can by simple differential calculus be expressed in the form (8), and equations of the form (11) will be generally covariant. Thus practically any conceivable law of gravitation would satisfy the condition of general covariance, and the true one is in no privileged position in this respect. Whether a law is true or not depends entirely on the values of the g's, which is the very point that the condition of general covariance has nothing whatever to say about.

Other conditions are, therefore, needed before any progress can be made towards determining the true law of gravitation. One of these is that at a great distance from any disturbing matter a particle behaves as if isolated, so that ds^2 tends to the value $c^2dt^2 - dx^2 - dy^2 - dz^2$. This, however, is still not enough, and to obtain a soluble problem Einstein has to introduce a further assumption whose correctness can be judged only by experience. The condition that ds^2 in its most general form can be transformed into the form $c^2dt^2 - dx^2 - dy^2 - dz^2$ can be shown to be that all the 96 components of a certain vector, the Riemann-Christoffel tensor, shall vanish, giving 20 independent conditions. Such a transformation is impossible in a gravitational field, for if it were possible the planets would move rigorously in straight lines and falling bodies on the earth could move with uniform velocity. Einstein, however, assumes that 6 relations between these components, which hold for isolated particles, still hold in a gravitational field. This renders the problem soluble, and the solution leads at once to the characteristic motions of the planets, including the advance of the perihelion of Mercury. There is, however, no *a priori* necessity about this hypothesis. It is essentially of the same character as the simple assumptions one ordinarily makes in mathematical physics when other evidence is lacking, and like them depends for its justification entirely on the verification of its consequences. If Mercury had been discovered

just before the theory was invented, no reliability could have been attached to the prediction that it would be subject to a perihelion advance of 43″ per century.

On the other hand, the planetary motions being known, it is possible to make considerable progress towards the determination of the form of ds, and the agreement of this with Einstein's form amounts practically to a proof of his hypothesis. We have the condition that at a great distance from gravitating matter ds^2 tends to $c^2dt^2 - dx^2 - dy^2 - dz^2$, or, if we take polar position coordinates, $c^2dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$. Consider now the field of a single gravitating particle. It is evident from conditions of symmetry that the g's must be independent of t and ϕ; the terms in $dr\, d\phi$ and $dr\, d\theta$ must be absent, for they would imply that if a particular path was possible, its image in one of its radii vectors would be impossible; those in $dr\, dt$, $d\theta\, dt$, $d\phi\, dt$ must also be absent, for they would imply that a path cannot be reversed; and $d\theta$ and $d\phi$ can only enter through $d\theta^2 + \sin^2\theta\, d\phi^2$, for any other form would imply differences in the motions of particles according to the direction we choose at random to measure θ from. Thus

$$ds^2 = c^2g_{44}\,dt^2 - g_{11}\,dr^2 - g_{22}\,r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \quad . \quad (12)$$

The g's must by symmetry be independent of θ; thus they are functions of r only and tend to unity when r tends to infinity. We can for convenience write now g_1 for g_{11}, etc.

A particle then moves so that $\delta\int ds$ would vanish for any small variation in the path. We can write this

$$\delta\int\{c^2g_4 - g_1\dot{r}^2 - g_2r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)\}^{\frac{1}{2}}\,dt = \delta\int \mathrm{L}\,dt, \text{ say, } = 0 \quad (13)$$

The equations of motion therefore take the form

$$\frac{d}{dt}\left(\frac{\partial \mathrm{L}}{\partial \dot{r}}\right) - \frac{\partial \mathrm{L}}{\partial r} = 0 \quad . \quad . \quad . \quad . \quad (14)$$

with three similar equations. We easily see, however, that if ϕ and $\dot{\phi}$ are zero initially they remain so permanently. We thus find for the equation of motion in θ

$$\frac{d}{dt}\frac{g_2r^2\dot{\theta}}{\mathrm{L}} = 0 \quad . \quad . \quad . \quad . \quad (15)$$

Also by multiplying the equations by $\dot{r}$ and $\dot{\theta}$ respectively and adding we find

$$\frac{d}{dt}\frac{g_4}{\mathrm{L}} = 0 \quad . \quad . \quad . \quad . \quad (16)$$

The two together give

$$\frac{g_2}{g_4}r^2\dot{\theta} = \text{const} = h \text{ say}, \quad . \quad . \quad . \quad (17)$$

so that

$$\frac{d}{dt} = \frac{h}{r^2}\frac{g_4}{g_2}\frac{d}{d\theta} \quad . \quad . \quad . \quad . \quad (18)$$

Then (16) gives

$$L^2/g_4^2 = \text{const} = A,$$

whence

$$c^2 g_4 - g_1\left(\frac{h}{r^2}\frac{g_4}{g_2}\frac{dr}{d\theta}\right)^2 - g^2\left(\frac{hg_4}{rg_2}\right)^2 = Ag_4^2 \quad . \quad . \quad (19)$$

Putting $1/r = u$ we have

$$\left(\frac{du}{d\theta}\right)^2 + \frac{g_2}{g_1}u^2 = \frac{c^2 g_2^2}{h^2 g_1 g_4} - \frac{Ag_2^2}{h^2 g_1} \quad . \quad . \quad . \quad (20)$$

The g's are not very different from unity; take then

$$g_1 = 1/(1+\Phi_1) \quad . \quad . \quad . \quad . \quad (21)$$

$$g_2 = 1+\Phi_2 \quad . \quad . \quad . \quad . \quad . \quad (22)$$

$$g_4 = 1/(1+\Phi_4) \quad . \quad . \quad . \quad . \quad (23)$$

where the Φ's are to be regarded as small quantities of the first order. Put also

$$A = c^2 - \alpha c^2 \quad . \quad . \quad . \quad . \quad (24)$$

Then

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{c^2}{h^2}(1+\Phi_2)^2(1+\Phi_1)(\alpha+\Phi_4) - u^2(\Phi_1+\Phi_2+\Phi_1\Phi_2) \quad (25)$$

This equation is exact. If now u is of order l, this shows that no term on the right except $\frac{c^2\alpha}{h^2}$ and $\frac{c^2\Phi_4}{h^2}$ is of order l^2; thus the largest terms in this equation are included in

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{c^2\Phi_4}{h^2} + \frac{c^2\alpha}{h^2} \quad . \quad . \quad . \quad (26)$$

The last term is a mere constant. Now we know that the planets move so that

$$u = l\{1 + e\cos p(\theta-\beta)\} \quad . \quad . \quad . \quad (27)$$

where p is very nearly unity and e is small. Substituting in (26) we see that the first term in Φ_4 must be equal to that of $2h^2lu/c^2$. Now by hypothesis Φ_4 is a function of u alone, involving no arbitrary constant. Hence h^2l must be the same for all planets; call it fM. Then to the first power of $1/r$

$$g_4 = 1/\left(1 + \frac{2fM}{c^2 r}\right) \quad . \quad . \quad . \quad . \quad (28)$$

Suppose now that the Φ's are expanded in ascending powers of u, so that

$$\Phi_1 = A_1 u + B_1 u^2 + \ldots \quad . \quad . \quad . \quad (29)$$

If, then, terms on the right of (25) involving u^2 and higher powers are neglected, we shall have

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \text{a linear function of } u,$$

and the solution will be of the form

$$u = l\{1 + e\cos(\theta - \beta)\}, \quad . \quad . \quad . \quad (30)$$

that is to say, a motion in a conic with no motion of the perihelion. The motion of the perihelion of Mercury must, therefore, arise from terms involving u^2 at least; and as the additional terms of lower orders contribute nothing to the distortion of the orbit, this will be correctly given if we consider only the terms

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{c^2}{h^2}\left\{\alpha - \frac{2fM}{c^2}u + \text{terms in } u^2 + \text{terms of order } \left(\frac{fM}{c^2}u\right)^3 - u^3(A_1 + A_2) + \text{terms in } u^4\right\} . \quad . \quad (31)$$

The ratio of the first portion of the terms in u^3 to the last is of order $\frac{f^3M^3}{c^4h^2} \div (A_1 + A_2)$; and $A_1 + A_2$, by analogy with A_4, is presumably of order $\frac{fM}{c^2}$. Also h^2 is of order $fM\alpha$, where α is the mean distance of the planet considered. Thus the ratio is of order $fM/c^2\alpha$, which is less than 10^{-7} for the nearest planet. It is also seen that $c^2\alpha/h^2$ is of order l^2, so that the terms involving αu^2 are of the second order, while the term in $(A_1 + A_2)u^3$ is of the first. Thus the only terms in u^2 we need retain on the right are those independent of α. Thus

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{c^2}{h^2}\{\alpha + A_4 u + B_4 u^2 + A_4(2A_2 + A_1)u^2\} - (A_1 + A_2)u^3 \quad (32)$$

The solution for Mercury is known by observation to be

$$u = l(1 + e\cos p\theta) \quad . \quad . \quad . \quad . \quad (33)$$

where

$$p = 1 - 3fMl/c^2 \quad . \quad . \quad . \quad . \quad (34)$$

Substituting in (32), expanding in cosines of multiples of $p\theta$, and picking out the coefficient of $\cos 2p\theta$, we find at once

$$p^2 - 1 = 3(A_1 + A_2)l - c^2h^{-2}\{B_4 + 2A_2A_4 + A_1A_4\} \quad . \quad (35)$$

Now, without loss of generality we can take $A_2 = 0$. This indicates merely that distances from the sun are defined as proportional to the reciprocals of the angles subtended at the sun

by the same small body. Thus by comparing (34) and (35) we have

$$\frac{6fMl}{c^2} = -3A_1 l + \frac{c^2}{h^2}\left\{\frac{2fM}{c^2}A_1 + B_4\right\},$$

or substituting the first approximation to h^2

$$B_4 - \frac{fM}{c^2}A_1 - 6\left(\frac{fM}{c^2}\right)^2 = 0 \quad . \quad . \quad . \quad (36)$$

The available data on the motions of the planets have now been used up, and a certain amount of arbitrariness remains in assigning values to A_1 and B_4.

Before deciding on the most probable values of these let us consider the Newtonian law. In this the ordinary theory shows that

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{c^2}{h^2}a + \frac{2fM}{h^2}u \quad . \quad . \quad . \quad (37)$$

and the motion of the perihelion is obviously zero. This is evidently satisfied in the simplest way if A_1 and B_4 are both zero, so that the Newtonian law is expressed by

$$ds^2 = c^2\left(1 + \frac{2fM}{c^2r}\right)^{-1}dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi) \quad . \quad (38)$$

Suppose now that in the actual law, satisfying equation (36), A_1 was zero. Then we should have

$$ds^2 = c^2\left(1 + \frac{2fM}{c^2r} + 6\left(\frac{fM}{c^2r}\right)^2\right)^{-1}dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \quad (39)$$

corresponding to a Newtonian potential

$$V = -\frac{fM}{r} + 2\left(\frac{fM}{c^2}\right)^2\frac{1}{r^3} \quad . \quad . \quad . \quad . \quad (40)$$

This satisfies the conditions indicated by planetary motions within the limits of observational error. It is not, however, the form adopted by Einstein. The first three coefficients follow no obvious law, so that it affords no reason to believe that g_4 is in fact a simple function. Einstein's solution is to take

$$B_4 = 4\left(\frac{fM}{c^2}\right)^2; \quad A_1 = -\frac{2fM}{c^2} \quad . \quad . \quad . \quad (41)$$

Thus

$$g_4 = \left\{1 + \frac{2fM}{c^2r} + 4\left(\frac{fM}{c^2r}\right)^2\right\}^{-1} = 1 - \frac{2fM}{c^2r} \quad . \quad . \quad (42)$$

within the limits of observational error.

The final solution is therefore

$$ds^2 = c^2\left(1 - \frac{2fM}{c^2r}\right)dt^2 - \left(1 - \frac{2fM}{r}\right)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \quad (43)$$

1. *The Influence of Gravitation on Light.*

The two alternative theories just given both agree with observation of the influence of matter on other matter; they say nothing about the influence of matter on light. A "velocity of light" appears in them, but only that appertaining to light where the quantity fM/c^2r is small; nothing is assumed about the behaviour of light when this number is appreciable. It may be possible, however, to find a hypothesis that will enable the movement of light to be predicted without introducing any further complexity than has already been introduced. We know that at a great distance light moves so as to keep $\int ds$ zero; though this by itself tells us little, since it does not specify the direction of motion. We also know, what is more useful, that the motion of light there is the limit of the motion of a particle when its velocity tends to c. For descriptive purposes, then, we will adopt the form of ds that would predict that a particle starting with the velocity of light would move in the way that light actually moves, if any such form is found possible. Suppose then

$$ds^2 = c^2 g_4 dt^2 - g_1 dr^2 - g_2 r^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \quad . \quad . \quad (1)$$

Then light moves so that

$$c^2 g_4 - g_1 \dot{r}^2 - g_2 r^2 \dot{\theta}^2 = A g_4^2 \quad . \quad . \quad . \quad (2)$$

and

$$\frac{g_2}{g_4} r^2 \dot{\theta} = h \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where A and h are as yet unknown.

Now at a great distance the velocity is c along a line passing at distance a from the centre of the sun, so that

$$\dot{r}^2 + r^2 \dot{\theta}^2 = c^2 \quad . \quad . \quad . \quad . \quad (4)$$

$$r^2 \dot{\theta} = ac \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Hence

$$A = 0, \quad h = ac \quad . \quad . \quad . \quad . \quad (6)$$

Also $\dot{r}$ vanishes when

$$\frac{r}{a} = \left(\frac{g_4}{g_2}\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (7)$$

The change in θ as r decreases from infinity to the point where $\dot{r}$ vanishes is

$$\int \frac{ac}{r^2} dt = \int_{a(g_4/g_2)^{\frac{1}{2}}}^{\infty} \frac{ac\, dr}{r^2 \dot{r}} \quad . \quad . \quad . \quad (8)$$

and in the notation of the last section this is

$$\tfrac{1}{2}\pi + \tfrac{1}{2}\frac{A_4 - A_1}{a} \quad . \quad . \quad . \quad . \quad (9)$$

The total deflection of a ray passing at distance a from the sun and away to the other side is, therefore, double this, or

$$\frac{A_4 - A_1}{a}. \quad . \quad . \quad . \quad . \quad (10)$$

Now, the result of the eclipse expeditions is that this deflection is $4fM/c^2a$, which shows that

$$A_4 - A_1 = \frac{4fM}{c^2} \quad . \quad . \quad . \quad . \quad (11)$$

If now we want a theory that will give an account of *both* objects moving with the velocity of light and planetary motions, this equation must be combined with those of the last section, namely,

$$A_4 = \frac{2fM}{c^2} \quad . \quad . \quad . \quad . \quad (12)$$

$$B_4 - \frac{fM}{c^2}A_1 - 6\left(\frac{fM}{c^2}\right)^2 = 0 \quad . \quad . \quad . \quad (13)$$

From these we deduce at once

$$A_1 = -\frac{2fM}{c^2}; \; B_4 = 4\left(\frac{fM}{c^2}\right)^2 \quad . \quad . \quad . \quad (14)$$

Thus the only form of ds that can give a correct account of the planetary motions and the displacement of star images is Einstein's.

III. Summary.

It has been shown from considerations depending on the observed luminosity of the zodiacal light and the solar corona that the matter between the orbit of Mercury and the surface of the sun is incompetent to produce any noticeable effect on the motions of the inner planets by its gravitation, or on the displacements of star images near the sun by its refraction. Accordingly the motion of the perihelion of Mercury and the displacements of star images are most naturally attributed to the gravitation of the sun. Incidentally estimates of the densities of the corona and the zodiacal matter are made.

The perihelion motion and the displacements of stellar images not being accounted for by the Newtonian form of the law of gravitation, a new law of gravitation is demanded. In this paper all laws of the form

$$\delta \int ds = 0,$$

where

$$ds^2 = \sum_i \sum_j g_{ij}\, dx_i\, dx_j$$

and δ denotes a small variation in the path of integration (a form which covers practically any hypothesis to which the principle of

least action could apply) are considered. It is found that the only form of ds^2 that is consistent with the observational data is Einstein's, namely

$$ds^2 = c^2\left(1 - \frac{2fM}{c^2r}\right)dt^2 - \left(1 - \frac{2fM}{c^2r}\right)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

The planetary motions could be accounted for by making

$$ds^2 = c^2\left\{1 - \frac{2fM}{c^2r} - 2\left(\frac{fM}{c^2r}\right)^2\right\}dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

but this only gives half the observed displacement of star images, so that light would obey a different law from matter, a result to be avoided on the ground of economy of hypotheses.

The spectral shifts are discussed, and it is decided that they afford no satisfactory evidence for or against the theory of gravitation. On the other hand, they do give fairly clear evidence that the assumption that the atom is a natural clock vibrating in the same period both in the laboratory and on the sun is incorrect.

On the Validity of the Principles of Relativity and Equivalence.
By Professor R. A. Sampson, F.R.S.

Einstein's theory of gravitation is offered for acceptance or rejection on the results of observed phenomena, and in this respect it is independent of the arguments by which it is derived. These arguments, however, are of great interest, especially the two axioms from which it starts, the Principle of Relativity and the Principle of Equivalence; their generality seems to convey a certain inevitableness in the conclusions from them, and therefore I offer the following short considerations as to their nature and limitation.

The Principle of Equivalence as given by Eddington * asserts that a "gravitational field of force is exactly equivalent to a field of force introduced by a transformation of the [space-time] coordinates of reference, so that by no possible experiment can we distinguish between them." Put otherwise, it is the attempt to separate what is personal in our perception of a law of nature from what is essential, by assuming that the latter is found in the residue which survives in invariant form after any permissible mathematical transformation. To seize exactly what is implied in such a step we may compare the complete and familiar transformation of conics into one another. Proceeding by substitution we find, for example, that the single property of a circle, which accounts for the equality of its radii, the perpendicular bisection of its chords, and other features by which we distinguish it from other ellipses is that every circle passes through two imaginary

* *Report on the Relativity Theory of Gravitation* (1918), p. 19.

fixed points at infinity—the focoids. If we are interested in what can be explored in the eminently real field of Nature, it is evident that this leads us away from reality, not closer to it. At the cost of being a little abstract it is worth stating why. To increase our knowledge of the circle, which is "given" as something special and therefore incomplete, we employ algebra, which is a machine and always works completely since its laws are complete; but algebra, made by the mind, cannot transcend the mind's limitations, and secures its completeness by inventing return paths to its starting point in the form of negative and imaginary regions of quantity which the mind can indeed understand and use, but which have no existence in the field of experience. The whole is consistent, however, and when it falls again into the real field, it cannot lead us to error. We may expect a similar situation to result from an algebraical application of the Principle of Equivalence. Now it is to be remarked that Nature is everywhere "given" in special limited forms. This is to many minds one of its chief fascinations. The forms of animated life, with their immense diversity, are extraordinarily specialised. The records of geology, natural chemical compounds, the solar system and universe of stars, all strike one by their limited exceptional character compared to the combinations that might be.

Without doubt the elements of which they are composed are subject to more general laws, but it is a pure assumption to suppose that they possess the generality of an algebra, which, for example, treats as irrelevant the difference between past and future and the whole movement of life in one direction, not to speak of introducing an imaginary time which exists neither in the domain of the past nor of the future. For the reasons stated above such an assumption is unobjectionable in that it cannot introduce error of prediction in any part of the real verifiable region of experience which we term Nature, but its success there carries with it no confirmation of what it implies outside this region. I ought to add that I am not aware of any illegitimate claim of this kind for the theory advanced by those familiar with it and qualified to speak, but to state the position may assist novices like myself to grasp more precisely what has been proved. Even some of the consequences within the real domain are so startling that it is difficult to believe they have any relationship to reality. Take, for example, the equation giving the velocity of light in the neighbourhood of a material particle or nucleus, given by Eddington (*Report*, p. 53),

$$v = \gamma(1 - (1 - \gamma)\sin^2 \mathrm{V})^{-\frac{1}{2}},$$

where V is the direction made with the radius vector and $\gamma = 1 - 2m/r$. Let the light move directly towards the nucleus so that $\mathrm{V} = 0$, and we see that it comes to a dead stop at the finite distance $r = 2m$ from the centre. This is the motion as estimated by an outside observer. An observer travelling with the light and using the associated "proper time" would by hypothesis

estimate the velocity differently and conclude that it was unchanged, *i.e.* unity. For the Sun, $r = 2m = 3$ km., and to concentrate the mass of the Sun within that volume is, shortly, an impossibility, but that does not seem to me to shake the impression conveyed by this inference, that we are not dealing with a more real description of Nature, but merely with a mathematical expression of undefined validity.

I should put it that to start with the principle as a mathematical postulate merely defines the domain of its own validity, but gives us no more assurance of the truth of its deductions. Its value none the less may be immense, because it is mathematical and therefore coherent in all its parts, so permitting us to bring into comparison deductions from widely separated lines of thought.

I would also offer a few remarks, some of which are by no means novel, on the Principle of Relativity, or the necessary self-elimination of any assumed absolute linear motions from the measures of phenomena. On the point of principle, however, linear motion cannot be treated differently from angular motion, and it is nowhere contended that angular motions eliminate themselves. Newton's remarks, in his celebrated *Scholium* on space and time, on to the phenomena of relative and absolute rest and rotation as illustrated by the water in a whirling bucket, still bear only the plain sense he attached to them—that we are able from the phenomena of rotation to detect something real in direction in space. For many years I regarded this conclusion as a glaring paradox, indicating perhaps something ill-defined in the associated idea of infinity; but it is difficult to see why. We are supposed by philosophers to have some relation ourselves, individually, to Reality, and the moral world is supposed to be arranged in grades of Values. These are not fanciful views, but the simplest conclusions upon which thinkers can rest. They are almost universally accepted. Why, then, should we rebut the idea that something definite in direction can be perceived in space, at any rate until we see some chance to argue it away? The interpretation of angular motion seems to me more urgent and more illuminating than even that of gravitation, as it is also simpler to those qualified to handle it. Meanwhile obscurities are carried into all fundamental work. It is impossible to say, for example, what is the meaning of the equations for the orbital motion of a particle about a gravitating nucleus (*cf.* Eddington, *Report*, p. 49), unless we know whether the datum for the azimuth of the particle is fixed or in motion "absolutely." I would illustrate my point further by the argument from which the "proper time" of a moving observer is inferred in the theory of relativity. So far as its zero departs from that of the fixed or standard observer, it is often imagined as transmitted from one to the other by a wireless signal, when the moving observer's clock will be "absolutely" slow upon the other by ux/c^2, where u is the velocity, x the separation, and c the velocity of light. Now, imagine this signal to be sent out from a point on the equator of the earth, and after passing all round the

earth in every direction to come back again to the signal point. If necessary we may imagine a perfectly reflecting screen surrounding the earth that would secure the passage. Now, with a rotating earth it seems hardly credible that all the issuing waves would come back at the same time and in identical phase to the starting point, which had moved in the interval, though this would be required if the Principle of Relativity applied to rotation. One would expect none at all to pass through it except those which went in two opposite directions along the equator, and these to arrive at different phases. But this, if it be so, would again permit us to mark the axis of rotation. It would also provide two different "proper times" for the moving point, one fast and the other slow. It would give two different measures for the time of transmission, E to W and W to E. The quantities involved are so minute that it seems unfortunately beyond hope to put it to a test by an organised chain of signal stations round the earth, but the theoretical difficulty is none the less clear.

I would remark in conclusion that it is easy to see how the Principle of Relativity should fail to verify itself for rotation and yet apply for translation. If we regard translation as a degenerate case of rotation in which the axis of rotation is infinitely remote, it is evident that the direction of this axis cannot be deduced from the single member of the couple, which is all that the known linear translation supplies; there is no definite axis to discover, associated with a translational motion, and hence if we try to discover one, algebraically or otherwise, the linear motions eliminate themselves altogether and we fail—and conclude from the failure the Principle of Relativity as far as concerns such motions.

Note on the Reduction of Latitude Variation Observations. By H. Spencer Jones, M.A., B.Sc.

(*Communicated by the Astronomer Royal.*)

1. *Introduction.*—In the reduction of the observations made with the floating zenith-telescope at the Royal Observatory, Greenwich (*M.N.*, **79**, 542, 1919), a rigid solution by least squares was adopted for the determination of the aberration constant and the latitude variation. This method of reduction apparently differs very considerably from that used in the reductions of the International Latitude Service (see *Resultate des Internationalen Breitendienstes*), which has been generally followed in the reduction of latitude-variation observations. In particular, the former method referred to appears to avoid the difficulty of deciding in what manner the closing-error should be divided amongst the several periods. Several inquiries having been received as to

the relationship of the two methods, the matter is dealt with in the present note.

2. *Observations.*—It will be assumed that the observations have been made according to a programme similar to that adopted in the International Latitude Service, and for simplicity it will further be supposed that the observations have extended over exactly one year.

The year is divided into twelve periods, during which twelve groups of stars are observed, the observations being made as nearly as possible symmetrically about midnight. In the first period groups 1 and 2 are observed, in the second period groups 2 and 3, and so on; the last period being groups 12 and 1. It will be assumed that the latitudes given by individual pairs in any given group have been compared, and corrections applied to reduce each pair to the mean of the group. From these corrected values, mean latitudes for each group and period are derived. We will deal with the residuals left by subtracting the adopted latitude for the place of observation from these mean latitudes.

In the first period the residuals from evening and morning observations are denoted by r_{1a}, r_{1b} respectively; in the second period by r_{2a}, r_{2b}, etc. Then, apart from accidental errors, the values of r_{sa}, r_{sb} should be equal.

If in the sth period the mean residuals are based respectively upon the observations of n_{sa}, n_{sb} pairs, then n_{sa}, n_{sb} can be used as a measure of the weight of the observations.

We will also denote the group corrections by z_1, z_2, . . . z_{12}.

3. *Normal Method.*—In the method of reduction normally used, the latitudes obtained from the two groups observed during a given period should agree, apart from accidental errors and group-corrections. Hence we have the system of equations:—

$$\left.\begin{aligned} z_1 - z_2 &= r_{1a} - r_{1b} \\ z_2 - z_3 &= r_{2a} - r_{2b} \\ \cdot \quad &\quad \cdot \quad \cdot \quad \cdot \\ z_{12} - z_1 &= r_{12a} - r_{12b}. \end{aligned}\right\}$$

The sum of the left-hand sides of these equations vanishes identically. The sum of the right-hand sides, however, is found to differ systematically from zero, and is called the "closing-error"; we shall denote it by E.

$$\mathrm{E} = \Sigma_1^{12} r_{sa} - \Sigma_1^{12} r_{sb}.$$

The method of reduction adopted is to divide the closing-error equally between the twelve equations, giving, *e.g.*,

$$z_s - z_{s+1} = r_{sa} - r_{sb} - \frac{\mathrm{E}}{12} \qquad (s = 1, 2, \ldots 12).$$

The latitude during any period is obtained from the weighted value of $(r - z)$ during the period. Thus we obtain

$$(n_{11,a}+n_{11,b})t_{11}=n_{11,a}(r_{11,a}-z_{11})+n_{11,b}(r_{11,b}-z_{12}),$$
$$(n_{12,a}+n_{12,b})t_{12}=n_{12,a}(r_{12,a}-z_{12})+n_{12,b}(r_{12,b}-z_{1}).$$

Substituting for z_{11} and z_1 in these equations in terms of z_{12}, we obtain on reduction :—

$$t_{11}=-z_{12}+r_{11,b}+\frac{n_{11,a}}{n_{11,a}+n_{11,b}}\frac{\mathrm{E}}{12},$$
$$t_{12}=-z_{12}+r_{12,a}-\frac{n_{12,b}}{n_{12,a}+n_{12,b}}\frac{\mathrm{E}}{12}.$$

Therefore

$$t_{11}-t_{12}=r_{11,b}-r_{12,a}+\frac{\mathrm{E}}{12}\left\{\frac{n_{11,a}}{n_{11,a}+n_{11,b}}+\frac{n_{12,b}}{n_{12,a}+n_{12,b}}\right\} \quad . \quad . \quad . \quad (1)$$

This equation gives the variation of the latitude from period 11 to period 12. Similar equations can be derived for the differences between any two consecutive t's and the variation of the latitude from any adopted latitude can therefore be determined. It remains to compare equation (1) with the corresponding equation derived by the "least-square" method.

4. *The Least-square Method.*—In this method equations are written down for each group and period thus :

$z_1+t_1=r_{1,a}$	weight, $n_{1,a}$;	$z_2+t_1=r_{1,b}$	weight, $n_{1,b}$
$z_2+t_2=r_{2,a}$	,, $n_{2,a}$;	$z_3+t_2=r_{2,b}$	,, $n_{2,b}$
. . .	. . .	. . .	
$z_{12}+t_{12}=r_{12a}$	,, $n_{12,a}$;	$z_1+t_{12}=r_{12,b}$	,, $n_{12,b}$.

Normal equations for the z's and t's are then derived, giving :—

Normal equations for the z's :—

$$\left.\begin{aligned}&(n_{1,a}+n_{12,b})z_1+n_{1,a}t_1+n_{12,b}t_{12}=n_{1,a}r_{1,a}+n_{12,b}r_{12,b}\\&(n_{2,a}+n_{1,b})z_2+n_{2,a}t_2+n_{1,b}t_1=n_{2,a}r_{2,a}+n_{1,b}r_{1,b}\\&\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\&(n_{12,a}+n_{11,b})z_{12}+n_{12,a}t_{12}+n_{11,b}t_{11}=n_{12,a}r_{12,a}+n_{11,b}r_{11,b}.\end{aligned}\right\}$$

Normal equations for the t's :—

$$\left.\begin{aligned}&(n_{1,a}+n_{1,b})t_1+n_{1,a}z_1+n_{1,b}z_2=n_{1,a}r_{1,a}+n_{1,b}r_{1,b}\\&(n_{2,a}+n_{2,b})t_2+n_{2,a}z_2+n_{2,b}z_3=n_{2,a}r_{2,a}+n_{2,b}r_{2,b}\\&\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\&(n_{12,a}+n_{12,b})t_{12}+n_{12,a}z_{12}+n_{12,b}z_1=n_{12,a}r_{12,a}+n_{12,b}r_{12,b}.\end{aligned}\right\}$$

The elimination of the z's gives equations containing the t's only, of which the first is :—

$$t_1\left\{\frac{n_{1,a}n_{12,b}}{n_{1,a}+n_{12,b}}+\frac{n_{2,a}n_{1,b}}{n_{2,a}+n_{1,b}}\right\}-\frac{n_{1,a}n_{12,b}}{n_{1,a}+n_{12,b}}t_{12}-\frac{n_{1,b}n_{2,a}}{n_{2,a}+n_{1,b}}t_2$$
$$=\frac{n_{1,a}n_{12,b}}{n_{1,a}+n_{12,b}}\{r_{1,a}-r_{12,b}\}-\frac{n_{2,a}n_{1,b}}{n_{2,a}+n_{1,b}}\{r_{2,a}-r_{1,b}\}$$

writing

$$w_{s,s-1} = n_{s,a} n_{s-1,b}/(n_{s,a} + n_{s-1,b})$$
$$\lambda_s = w_{s,s-1}\{r_{s,a} - r_{s-1,b}\} - w_{s+1,s}\{r_{s+1,a} - r_{s,b}\}$$

this becomes

$$(w_{1,12} + w_{2,1})t_1 - w_{1,12}t_{12} - w_{2,1}t_2 = \lambda_1.$$

Similarly, 11 other equations can be written down. These equations are not independent, as their sum vanishes identically. This is because only differences between the t's can be determined. The actual values of the t's depend upon the adopted mean latitude. The equations possess the further property that the sum of the coefficients on the left-hand side of any equation is zero. These properties are useful as checks in solving the series.

A straightforward solution gives for $(t_{11} - t_{12})$:—

$$t_{11} - t_{12} = \frac{w_{11,12}^{-1}}{w_{12,11}^{-1} + w_{11,10}^{-1} + \ldots + w_{1,12}^{-1}} \{\lambda_{11}(w_{11,10}^{-1} + w_{10,9}^{-1} + \ldots + w_{1,12}^{-1}) + \ldots + \lambda_s(w_{s,s-1}^{-1} + \ldots + w_{1,12}^{-1}) + \ldots + \lambda_1 w_{1,12}^{-1}\}$$

and the values for $(t_1 - t_2)$, etc., may be written down from symmetry.

The expression in the bracket on the right-hand side may be simplified by substituting the values of the λ's, and is easily found to give the value

$$r_{1,a} - r_{12,b} + r_{2,a} - r_{1,b} + \ldots + r_{12,a} - r_{11,b} - w_{12,11}\{w_{12,11}^{-1} + w_{11,10}^{-1} + \ldots + w_{1,12}^{-1}\}(r_{12,a} - r_{11,b}) = \mathrm{E} - w_{12,11}(w_{12,11}^{-1} + \ldots + w_{1,12}^{-1})(r_{12,a} - r_{11,b})$$

where E is the expression previously denoted the "closing-error."

Further

$$w_{s,s-1}^{-1} = n_{s,a}^{-1} + n_{s-1,b}^{-1}.$$

Therefore

$$t_{11} - t_{12} = \frac{n_{12,a}^{-1} + n_{11,b}^{-1}}{\Sigma_1^{12} n_{s,a} + \Sigma_1^{12} n_{s,b}} \left\{ \mathrm{E} - \frac{\Sigma_1^{12} n_{s,a} + \Sigma_1^{12} n_{s,b}}{n_{12,a}^{-1} + n_{11,b}^{-1}} (r_{12,a} - r_{11,b}) \right\}$$

$$= r_{11,b} - r_{12,a} + \mathrm{E} \cdot \frac{n_{12,a}^{-1} + n_{11,b}^{-1}}{\Sigma_1^{12} n_{s,a} + \Sigma_1^{12} n_{s,b}} \quad . \quad . \quad . \quad (2)$$

5. The expressions (1), (2) above for $(t_{11} - t_{12})$ are to be compared. It will be seen that they differ only in the coefficient of the closing-error E. If, however, the weight of each group and period is the same, both expressions reduce to

$$t_{11} - t_{12} = r_{11,b} - r_{12,a} + \frac{\mathrm{E}}{12},$$

and the two methods of reduction would therefore agree in giving identical values for the latitude-variation. The two methods would also agree if all the evening groups had equal weight and the morning groups a different equal weight. In no other case will the reductions agree. The difference will become more marked, the greater the inequalities in the numbers of pairs observed during different periods.

In the normal method of reduction the method of partitioning the closing-error has always been wrapped up in some obscurity, arising from its cause being unknown. It was at first attributed to aberration, but this is now known not to be correct. The least-square method of solution leaves no arbitrariness in the way in which it should be partitioned, and undoubtedly must be regarded as providing a more satisfactory method of dealing with the term.

6. The necessary modification in the method of treating the closing-error in order to make the two methods of reduction agree is easily indicated.

The normal equation for z_1, derived from the equations in § 4 is easily seen to be :—

$$z_1\left\{\frac{n_{1,a}n_{1,b}}{n_{1,a}+n_{1,b}}+\frac{n_{12,a}n_{12,b}}{n_{12,a}+n_{12,b}}\right\}-\frac{n_{1,a}n_{1,b}}{n_{1,a}+n_{1,b}}z_2-\frac{n_{12,a}n_{12,b}}{n_{12,a}+n_{12,b}}z_{12}$$
$$=\frac{n_{1,a}n_{1,b}}{n_{1,a}+n_{1,b}}(r_{1,a}-r_{1,b})-\frac{n_{12,a}n_{12,b}}{n_{12,a}+n_{12,b}}(r_{12,a}-r_{12,b}).$$

This may be solved as in the case of the corresponding l equation, or more directly thus :—
Writing

$$n_{1,a}n_{1,b}/(n_{1,a}+n_{1,b})=w_1, \text{ etc., we have}$$
$$w_1(z_1-z_2-r_{1,a}+r_{1,b})=w_{12}(z_{12}-z_1-r_{12,a}+r_{12,b})$$

or

$$\frac{z_1-z_2-r_{1,a}+r_{1,b}}{w_1^{-1}}=\frac{z_{12}-z_1-r_{12,a}+r_{12,b}}{w_{12}^{-1}}=\ldots$$
$$=\frac{-\mathrm{E}}{\Sigma w^{-1}}.$$

Therefore

$$z_1-z_2=r_{1,a}-r_{1,b}-\frac{n_{1,a}^{-1}+n_{1,b}^{-1}}{\Sigma n_{s,a}^{-1}+\Sigma n_{s,b}^{-1}}\mathrm{E} \quad . \quad . \quad . \quad (3)$$

The normal method of reduction, on the other hand, gives

$$z_1-z_2=r_{1,a}-r_{1,b}-\frac{1}{12}\mathrm{E} \quad . \quad . \quad . \quad . \quad (4)$$

The method of partition of the closing-error required by (3), instead of the equal division required by (4), will lead to the

values of the latitude-variation given by the least-square method of solution. The weighted division of the closing-error indicated in (3) is easily arranged, and the method can be extended without difficulty to results derived from observations extending over more than one year.

7. *Conclusion.*—The two methods of reduction differ only in the method of disposing of the closing-error. Provided that the observations are fairly equally distributed throughout the year, the results given by both methods will be in agreement. With unequally distributed observations, however, the divergency between the two methods may become important, since the closing-error is normally about $-0''{\cdot}2$ or $-0''{\cdot}3$. In the absence of knowledge as to the physical cause of the closing-error, the least-square solution must be regarded as the more satisfactory. The method of partitioning the closing-error, which brings the results by the two methods into agreement, is indicated.

Statistical Investigation of the Distribution of the Stars and their Luminosities. By J. Halm, Ph.D.

§ 1. *The Problem of Stellar Statistics and General Mathematical Theory.*

The object of stellar statistics is the establishment of the law according to which the stars are distributed throughout space in regard to numbers and luminosities. This object is attained when once the mathematical character of a single function is known, which will be denoted as the distribution function and which is defined in the following manner:—

Let a solid cone be described with angle $d\omega$, whose apex is at the sun and whose axis lies in the direction of a point P fixed by the distance r and the spherical co-ordinates λ, β. The number of stars inside this cone which are situated between the distances $(r \pm \frac{1}{2}dr)$, and whose *apparent* magnitudes lie between $(m \pm \frac{1}{2}dm)$ may generally be expressed in the form

$$dN = f(r, m, \lambda, \beta)\, r^2\, dr\, dm\, d\omega \quad . \quad . \quad . \quad (1)$$

where f is the distribution-function expressed in terms of r, m, λ, and β. Obviously, since the *absolute* magnitude M, *i.e.* the magnitude which would be observed if the star were transferred from distance r to distance 1 parsec., is defined by

$$M = m - 5 \log r \quad . \quad . \quad . \quad . \quad (2)$$

the number dN can also be expressed in the form

$$\left.\begin{aligned} dN &= F(r, M, \lambda, \beta)\, r^2\, dr\, dm\, d\omega \\ &= F(r, m - 5 \log r, \lambda, \beta)\, r^2\, dr\, dm\, d\omega \end{aligned}\right\} \quad . \quad . \quad (3)$$

If the mathematical form of the function F is known, then for any cone $d\omega$ the following data may be calculated:—

(*a*) the number of stars whose apparent magnitude is brighter than m:

$$N_m = \int_0^\infty r^2\,dr \int_{-\infty}^{m} F\,dm\,d\omega \quad . \quad . \quad . \quad (4)$$

(*b*) the mean parallaxes of stars of given magnitude m:

$$\pi_m = \frac{\int_0^\infty Fr\,dr}{\int_0^\infty Fr^2\,dr} \quad . \quad . \quad . \quad . \quad (5)$$

(*c*) The number of stars between specified limits of distance $(r \pm \frac{1}{2}\Delta r)$ and apparent magnitude $(m \pm \frac{1}{2}\Delta m)$:

$$\Delta N_{r,m} = Fr^2\,\Delta r\,\Delta m \quad . \quad . \quad . \quad . \quad (6)$$

Conversely, if observations supply data concerning the relations *a*, *b*, *c*, the problem is to find a mathematical expression for the function F which satisfies the given data.

The magnitude of a star, as determined by photometric observations, depends on the mean wave-length, l, of that portion of the spectrum which affects the receiving apparatus. It is necessary, therefore, that the function F should satisfy the observed relations *a*, *b*, *c* under all conditions of wave-length. In practice magnitudes are determined within the range of wave-lengths affecting the retina of the eye (visual magnitudes) on the one hand, and within the range affecting the photographic plate (photographic magnitudes), on the other. The function F must therefore satisfy the observed conditions *a*, *b*, *c* for both the visual and photographic scale of magnitudes. In its most general form it must thus be defined as a function of the five parameters:

$$r, \mathbf{M}, \lambda, \beta, l,$$

the parameter l being implicitly involved in the parameter M.

Since one particular form of the function F will be of special importance in the subsequent discussion, some mathematical conclusions derived from it may be discussed here for further reference. Let F be assumed to be of the form

$$F[\alpha r, h(M + A)] \quad . \quad . \quad . \quad . \quad (7)$$

where α and A are functions of the spherical co-ordinates λ, β only, and h represents a constant. In place of (3) we write now

$$dN = F[\alpha r, h(m - 5\log r + A)]r^2\,dr\,dm\,d\omega. \quad . \quad (8)$$

α and A having constant values throughout any one of the cones $d\omega$, but changing from one cone to another. For a second cone of the same aperture $d\omega$ let equation (8) be expressed by

$$dN' = F[\alpha' r', h(m' - 5\log r' + A')]\,r'^2\,dr'\,dm'\,d\omega \quad . \quad (9)$$

which may be written in the form

$$(\alpha'/\alpha)^3 d\mathrm{N}' = \mathrm{F}[\alpha' r',\ h(m' - 5\log r' + \mathrm{A}')]\left(\frac{\alpha' r'}{\alpha}\right)^2 d\left(\frac{\alpha' r'}{\alpha}\right) dm'\, d\omega \quad (10)$$

Comparing (8) and (10), the function F has the same value in both cones when

$$\alpha r = \alpha' r';\ m' = m + 5\log(\alpha/\alpha') + (\mathrm{A} - \mathrm{A}') \quad . \quad . \quad (11)$$

Writing

$$\Delta m = 5\log \alpha/\alpha' + (\mathrm{A} - \mathrm{A}'),$$

by integrating (8) between the limits $r = 0, \infty$ and $m = -\infty$, $m + \Delta m$, and (10) between $r' = 0, \infty$ and $m' = -\infty, m'$, we find

$$\log \mathrm{N}_{m+\Delta m} = \log \mathrm{N}'_{m'} + 3\log \alpha'/\alpha \quad . \quad . \quad (12)$$

Further, from equation (5), we obtain a similar relation between the mean parallaxes, viz.

$$\log \pi_{m+\Delta m} = \log \pi'_{m'} - \log \alpha'/\alpha \quad . \quad . \quad . \quad (13)$$

Suppose now that the relations between m', $\mathrm{N}'_{m'}$, and $\pi'_{m'}$ are known for a certain cone, then the relations between m, N_m, and π_m for any other cone are found by multiplying $\mathrm{N}'_{m'}$ into $(\alpha'/\alpha)^3$, $\pi'_{m'}$ into α/α' and replacing m' by $m' - \Delta m$,
where

$$\Delta m = 5\log \alpha/\alpha' + (\mathrm{A} - \mathrm{A}').$$

Conversely, if statistical observations show that the stars of the cosmic system surrounding our sun satisfy the conditions (12) and (13), we conclude that the distribution function has the form (7), viz.

$$\mathrm{F}[\alpha r,\ h(\mathrm{M} + \mathrm{A})],$$

where h is a constant for all cones, whereas α and A are generally functions of the spherical co-ordinates λ, β.

If curves are constructed for different cones with the co-ordinates m and $\log \mathrm{N}$, these curves can be brought into coincidence with the curve $(m', \log \mathrm{N}')$ of any specially selected cone by appropriate shifts parallel to the axes of co-ordinates, the amounts of these shifts being represented by $(5\log \alpha/\alpha' + \mathrm{A} - \mathrm{A}')$ and $3\log \alpha/\alpha'$ respectively.

At the same time, if curves are drawn with the co-ordinates m and $\log \pi_m$, the curves can be brought into coincidence with the curve $(m', \log \pi'_{m'})$ by shifts along the co-ordinates whose amounts are represented by $(5\log \alpha/\alpha' + \mathrm{A} - \mathrm{A}')$ and $\log \alpha'/\alpha$.

We shall see that the stellar system, so far as present observations are able to testify, satisfies these conditions.

A particular case of the distribution-function (7) which underlies the statistical theories of Kapteyn and Schwarzschild is expressible in the form

$$\mathrm{F} = \phi(\alpha r)\psi[h(\mathrm{M} + \mathrm{A})] \quad . \quad . \quad . \quad (14)$$

The distribution-function within a given cone is represented by the product of two functions, the one depending on the distance only, the other on the absolute magnitudes only. The function $\phi(\alpha r)$ represents in this case the total number of stars contained in unit volume at distance r, and for this reason is denoted as the *density*-law. The function ψ, multiplied into dM, represents the fraction of the total number with absolute magnitudes between the limits $(\mathrm{M} \pm \frac{1}{2}d\mathrm{M})$. Hence

$$\int_{-\infty}^{+\infty} \psi d\mathrm{M} = 1.$$

Obviously, if (14) represents the true form of the distribution-function, the factor α cannot depend on the effective wave-length for which the magnitudes are determined. Hence the value of α found for a given cone from the observed statistical material by means of the relations (12) and (13) must be the same irrespective of whether it has been obtained from material based on the visual or photographic magnitudes.

The special form of F represented in equation (14) can be accepted only when this condition is satisfied. This establishes an important criterion by means of which the Kapteyn-Schwarzschild theory can be tested.

The subsequent discussion is intended, in the first place, to show that the observed relations (m, N) and (m, π_m) satisfy the conditions expressed in equations (12) and (13). It will be demonstrated, in particular, that the curves $(m, \log \mathrm{N})$ obtained for different cones, whether based on photographic or visual magnitudes, can be brought into coincidence with one single standard curve $(m', \log \mathrm{N}')$ by adding appropriate constants to the m- and $\log \mathrm{N}$-co-ordinates which have different values for different cones. From these constants the quantities $\log \alpha'/\alpha$ and $(\mathrm{A}' - \mathrm{A})$ may be calculated by means of equations (11) and (12).

Having demonstrated the distribution-function F to be of the general form (7), an attempt is made to define F analytically in such a manner that all the results of statistical observations at our disposal can be represented. These observations comprise :—

1. The relation between apparent magnitude (photographic and visual) and star numbers in different galactic latitudes.

2. The relation between apparent visual magnitude and mean parallax in different galactic latitudes.

3. The distribution in distance of stars of given apparent magnitude (according to Dyson and Eddington).

§ 2. *The Observed Relation between Star-Counts and Apparent Magnitudes.*

The method of observation consists in counting within a given angular field, usually one square degree, the number of stars brighter than specified limiting magnitudes which are determined by fundamental photometric observations. According to whether

these magnitudes refer to the portion of the star's total radiation which affects the retina of the eye or to that which reacts on the photographic plate, we distinguish between *visual* and *photographic* magnitudes. Since the ratio between the visible and photographic intensities depends on physical conditions inherent in the stars themselves, *i.e.* on their present stage of evolution, and since there is no *a priori* reason why these conditions should be independent of the positional co-ordinates, it is clearly necessary to differentiate between star counts based on visual and photographic magnitudes. It is further of the highest importance that the magnitudes should express as correctly as possible the true light ratios, and therefore should be determined by strictly photometric methods. Before the relation between star numbers and magnitudes can be examined, a critical discussion of the limiting magnitudes adopted in the various statistical investigations is therefore desirable.

(*a*) *Discussion of the Limiting Magnitudes.*—As far as the visual magnitudes are concerned, the Harvard system appears to represent the closest approximation to the correct light ratios at present available. Its consistency with other photometric systems in the region of the brighter stars has been amply demonstrated. Recent investigations of the photo-visual magnitudes of faint stars have further proved the systematic correctness of the Harvard visual scale over a considerable range of the fainter magnitudes, probably down to the 18th. On these grounds, the general adoption of the Harvard visual system for the limiting magnitudes of star-counts is justified.

With regard to the "best" photographic system there exists, however, still some divergence of opinion. The limiting photographic magnitudes employed in statistical investigations have been standardised by direct photographic comparison with a selected sequence of stars in the neighbourhood of the northern pole whose magnitudes were previously determined by strict photometric methods. The accuracy of the photographic system depends thus entirely on the determination of the magnitudes of the polar standard stars.

A provisional list of such standard magnitudes has been published in Harvard Circular No. 170, and has been extensively used in recent statistical investigations, notably in those of Chapman and Melotte.

Systematic corrections to this provisional system were subsequently determined in a detailed examination of an extensive photometric material collected at the Harvard Observatory and published in vol. lxxi. of the *Harvard Annals* (table lxxv. on p. 224).

On the other hand, an entirely independent standardisation of the northern polar stars contained in H.C. 170 was undertaken at the Mount Wilson Observatory by Seares, the results being published in various papers of the *Mount Wilson Contributions*.

The, by no means inconsiderable, differences between the two systems Harvard 71 and Mount Wilson are shown in Table I.

Table I.

H. 71.	Mt. W.	H. 71.	Mt. W.	H. 71.	Mt. W.
4·00	3·94	9·00	9·12	14·00	14·24
5·00	4·95	10·00	10·18	15·00	15·24
6·00	5·97	11·00	11·24	16·00	16·24
7·00	6·99	12·00	12·24	17·00	17·24
8·00	8·05	13·00	13·24	18·00	18·24

In the region of the bright stars ($m<7$) the two systems agree fairly well both in zero-point and scale. On the other hand, in the region of the very faint stars ($m>11$) the scales agree, but the zero-points differ by $0^m\cdot24$. The intermediate interval (7^m to 11^m) is characterised by a very pronounced difference in scale, 1^m (Harvard) being equivalent to $1^m\cdot06$ (Mount Wilson).

An explanation of this somewhat serious discrepancy may perhaps be found in the method by which the final Harvard system has been derived. The method of reduction adopted in the Harvard investigation led originally to a system consistent with H.C. 170. Subsequently, this provisional system was compared with extraneous systems obtained at Göttingen, Yerkes, and Mount Wilson (H. 71, table lxx. p. 215) and the following differences were found (the Yerkes scale being reduced in the ratio 0·94 : 1 according to Schwarzschild and Parkhurst).

		m	m		
Göttingen,	H.C. 170 : Range :	4·4	7·5;	+0·05	+0·074($m-6\cdot0$)
Yerkes,	H.C. 170 :	4·4	7·5;	0·00	+0·043($m-6\cdot0$)
Mt. Wilson,	H.C. 170 :	3·3	10·4;	0·00	+0·066($m-6\cdot0$).

These figures show conclusively that in the region of the brighter stars the provisional system H.C. 170 requires a considerable correction in scale, a fact which was recognised by the Harvard investigators. By a somewhat doubtful combination of visual and photographic comparisons they derived the final correction to the H.C. system :

$$+0\cdot08+0\cdot050(m-6\cdot0).$$

The important point, however, which has not received sufficient attention in the Harvard discussion, is the question within what range of magnitudes this scale correction is required. That it cannot be carried beyond the 11th magnitude is shown by the fact that the scale of H.C. 170 agrees with the scale of Mount Wilson in the interval 11^m-18^m. Obviously guided by the evidence of the Göttingen and Yerkes series, which are limited to magnitudes brighter than $7^m\cdot5$, the conclusion was reached that a scale correction of the amount +0·05 should be applied only to magnitudes brighter than 7, and that beyond the 7th magnitude the scale correction should be zero. But this conclusion is emphatically negatived by the Mount Wilson observations which

show that the scale correction should be applied down to the 11th magnitude.

According to the combined evidence of the Göttingen, Yerkes, and Mount Wilson observations the difference (standard photographic − H.C. 170) should be

$$+0^{m}\cdot07+0^{m}\cdot061(m-6^{m}\cdot0)$$

for magnitudes brighter than 11, the correction to zero-point being chosen so as to bring the standard system into agreement with Mount Wilson. We thus find the following differences:—

m (ptg.).	(H. 71 − H.C. 170.)	(Std. ptg. − H.C. 170.)	(Std. ptg. − H. 71.)
4·5	+0·01	−0·02	−0·03
5·0	+0·03	+0·01	−0·02
5·5	+0·06	+0·04	−0·02
6·0	+0·08	+0·07	−0·01
6·5	+0·11	+0·10	−0·01
7·0	+0·13	+0·13	0·00
8·0	+0·13	+0·19	+0·06
9·0	+0·13	+0·25	+0·12
10·0	+0·13	+0·31	+0·18
11·0	+0·14	+0·38	+0·24
12·0	+0·14	+0·38	+0·24
13·0	+0·14	+0·38	+0·24

The differences of the last column agree almost exactly with the differences (Mount Wilson—H. 71) of Table I., showing that the Mount Wilson system agrees with the standard photographic system derived from the combined evidence of all the standardisations at present available.

In the following discussion of the relation between star-counts and limiting magnitudes, the latter will therefore be referred either to the *photographic* system of Mount Wilson or to the *visual* system of Harvard.

(*b*) *The Observed Relation between Star-Counts and Photographic Magnitudes.*—Of recent years important investigations regarding the relation between star-counts and magnitudes have been made by Kapteyn and Seares based on visual magnitudes, and by Chapman and Melotte and van Rhijn based on photographic magnitudes. These investigations furnish material for examining the relation between counts and magnitudes in reference to the galactic latitudes of the regions examined. The published records, however, refer only to averages taken over galactic parallels, and hence do not permit of examining the material in reference to longitude.

The present discussion, so far as *photographic* magnitudes are concerned, will be confined to the results published by van Rhijn in *M.N.*, 78, and more particularly to table ii. on p. 139 of his

paper which shows the values of log N (N being the number of stars per square degree brighter than a given limiting magnitude) for magnitudes 7·0 to 16·0 (on the system H. 71) and for 12 galactic parallels ranging from $\beta = 0°$ to $\beta = 90°$.

Corresponding values of m and log N, in any of the specified galactic parallels, may be considered as the co-ordinates of a curve representing the functional relation between m and N, whose geometrical character depends on the galactic position. If m is chosen as abscissa and log N as ordinate, the curves thus constructed are much steeper in low than in high latitudes. Closer examination, however, reveals a common property among these curves which is of considerable theoretical significance. It appears that any individual curve represents a branch of one and the same common curve, whose co-ordinates may be denoted by m' and log N′, and may be obtained from the latter by simply changing the zero-points of the co-ordinates. *Vice versa*, all the curves constructed from van Rhijn's table can be brought into one single curve by changing the m and log N co-ordinates into m' and log N′ through adding the corrections $(m' - m)$ and log N′/N shown in Table II.

TABLE II.

β.	Photographic Magnitudes. $(m'-m)$.	log N′/N.
0°	−1·30	−0·78
10	−1·15	−0·69
20	−0·63	−0·37
30	+0·01	+0·03
40	+0·51	+0·33
50	+0·75	+0·51
60	+1·00	+0·67
70	+1·18	+0·77
80	+1·29	+0·84
90	+1·32	+0·86

The co-ordinates of the common curve (m', log N′) are given in Table III.

TABLE III.

m'.	log N′.	m'.	log N′.	m'.	log N′.
2·5	−2·951	8·5	+0·023	14·5	+2·302
3·5	−2·438	9·5	+0·466	15·5	+2·596
4·5	−1·928	10·5	+0·883	16·5	+2·868
5·5	−1·425	11·5	+1·276	17·5	+3·119
6·5	−0·932	12·5	+1·643	18·5	+3·351
7·5	−0·447	13·5	+1·985	19·5	+3·564

The data supplied in Tables II. and III. are sufficient to find the

numbers of stars (per square degree) in any galactic latitude β for any given limiting magnitude (photographic).

As an example let us calculate the star numbers for limiting photographic magnitude 13·00 (H. 71) or 13.24 (Mount Wilson) and compare with the observed figures of table ii. in van Rhijn's paper. The successive steps of this calculation are shown in the vertical columns of Table IV. They lead to the computed values of log N in the 7th column, which agree well with van Rhijn's values shown in column 8.

The inverse problem of finding the limiting magnitude when log N is given can of course be treated in an exactly similar manner by starting from column 7 in the inverse order back to column 2, the values of m thus computed being shown in the last vertical column.

TABLE IV.

β.	m.	$(m'-m)$. Table II.	m'.	log N'. (Table III.).	log N'/N. (Table II.).	log N. (Compd.).	log N. (van Rhijn).	m. (Compd.).
0°	13·24	−1·30	11·94	1·44	−0·78	2·22	2·22	13·24
10	13·24	−1·15	12·09	1·50	−0·69	2·19	2·19	13·25
20	13·24	−0·63	12·61	1·68	−0·37	2·05	2·06	13·26
30	13·24	+0·01	13·25	1·90	+0·03	1·87	1·88	13·26
40	13·24	+0·51	13·75	2·07	+0·33	1·74	1·74	13·25
50	13·24	+0·75	13·99	2·14	+0·51	1·63	1·61	13·18
60	13·24	+1·00	14·24	2·22	+0·67	1·55	1·56	13·27
70	13·24	+1·18	14·42	2·28	+0·77	1·51	1·51	13·25
80	13·24	+1·29	14·53	2·31	+0·84	1·47	1·48	13·27
90	13·24	+1·32	14·56	2·32	+0·86	1·46	1·47	13·27

If any other limiting magnitude had been chosen, the agreement between the observed and calculated values of log N would have been found equally good. It must, however, be noted that van Rhijn's figures do not represent the actually observed star-counts (which are shown in his table i.) but values obtained from a smoothed curve. Had the actual observations been compared, the *accidental* discrepancies would of course appear larger, but the *systematic* agreement between the observed and calculated values would not have been affected. Our result amounts to this: If on a number of photographic plates containing regions of the sky situated in the same galactic latitude all the stars are counted whose photographic images exceed a certain fixed size and which are contained within one square degree on each plate, the limiting magnitude corresponding with the adopted star image can be found from the data given in Tables II. and III. Thus the limiting photographic magnitude of an astrographic catalogue can be ascertained provided the catalogue represents a complete list of the stars down to the faintest star image.

As an example, in the Cape Astrographic Zones (−41° to

-51° Dec.) the total numbers of stars were counted on three groups of plates, whose mean galactic latitudes were 70°, 40°, and 17° respectively, and the following values of log N were found :

β.	log N.	No. of Plates.
70°	1·63	189
40	1·85	189
17	2·25	126

With the aid of Tables II. and III. we find :

β.	log N.	log N′/N.	log N′.	m'.	$(m'-m)$.	m.
70°	1·63	+0·77	2·40	14·81	+1·18	13·63
40	1·85	+0·33	2·18	14·11	+0·51	13·60
17	2·25	−0·47	1·78	12·88	−0·80	13·68

The limiting magnitude of the Cape Astrographic Zones is thus 13·64, which agrees well with the value 13·7 found from photometric determinations.

In order to show more generally how far the photographic magnitudes derived from the star-counts by means of Tables II. and III. represent the standard photographic system (Mount Wilson) the above method has been applied to the entire material published by van Rhijn in the paper quoted. The magnitudes calculated from his star-counts, representing the mean for all galactic latitudes, are shown in the third column of Table V. and are compared with the photographic system adopted by van Rhijn (1st column) and with the Mount Wilson system (2nd column).

TABLE V.

Adopted.		Limiting Photographic Magnitudes. Mt. Wilson.	From Star-Counts.
4·00	visual	...	4·00
4·50	Harvard	...	4·45
5·00	+corrn. for colour	...	4·99
5·50		...	5·54
7·00	H. 71	6·99	7·00
8·00		8·05	8·05
9·00		9·12	9·10
10·00		10·18	10·16
11·00		11·24	11·23
12·00		12·24	12·26
13·00		13·24	13·25
14·00		14·24	14·22
15·00		15·24	15·23
16·00		16·24	16·25

(*c*) *The Star-Counts of Chapman and Melotte.*—A valuable test of the conclusions reached in the preceding section is afforded by the

investigations of Chapman and Melotte published in *Mem. R.A.S.*, **60**, part 4, and in *M.N.*, **78**, No. 1. The material taken from the former paper consists in the counts of stars brighter than the 9th magnitude from plates taken with the Franklin-Adams 6-inch lens, while the latter contribution supplies the revised counts of stars between the 12th and 16th magnitudes of the northern Franklin-Adams charts, and in addition an independent series of counts between the 11th and 13th magnitudes.

The limiting magnitudes in all these cases have been standardised by means of the north-polar sequence and are based on the provisional system H.C. 170. As before, the observed star counts were utilised in calculating the limiting magnitudes by means of Tables II. and III. Table VI. shows these calculated magnitudes for the individual galactic regions, and also the observed magnitudes on the adopted system H.C. 170, and the standard photographic system of Mount Wilson.

TABLE VI.

Limiting Magnitudes Calculated from the Star-Counts of Chapman and Melotte.

I. Franklin-Adams 6-inch lens.

Obsd. Magnitudes.		Calculated Magnitudes.				Mean.	Mt. W., – Compd.
H.C. 170.	Mt. Wilson.	$\beta=15°$.	25°.	35°.	45°.		
6·50	6·60	6·62	6·57	6·59	6·61	6·60	0·00
7·00	7·13	7·31	7·12	7·13	7·15	7·18	−0·05
7·50	7·66	7·76	7·68	7·72	7·64	7·70	−0·04
8·00	8·19	8·20	8·20	8·15	8·23	8·19	0·00
8·50	8·72	8·63	8·69	8·68	8·88	8·72	0·00
9·00	9·25	9·08	9·20	9·18	9·36	9·21	+0·04

II. Greenwich Series (*M.N.*, **78**, Table E).

Obsd. Magnitudes.		Calculated Magnitudes.			Mean.	Mt. W., Compd.
H.C. 170.	Mt. Wilson.	β–(0°–20°).	(20°–40°).	(40°–90°).		
11·00	11·38	11·47	11·37	11·33	11·39	−0·01
11·50	11·88	11·86	11·94	11·89	11·90	−0·02
12·00	12·38	12·37	12·29	12·30	12·32	+0·06
12·50	12·88	12·84	12·74	12·79	12·79	+0·09
13·00	13·38	13·44	13·32	13·32	13·36	+0·02

III. Franklin-Adams Charts (*M.N.*, **78**, Table A).

Obsd. Magnitudes.		Calculated Magnitudes.								Mean.	Mt. W. Compd.
H.C. 170.	Mt.W.	$\beta=5°$.	15°.	25°.	35°.	45°.	55°.	65°.	90°.		
12·00	12·38	12·65	12·51	12·63	12·89	12·64	12·73	12·09	12·65	12·60	−0·2
13·00	13·38	13·45	13·30	13·55	13·81	13·57	13·58	13·63	13·63	13·57	−0·1
14·00	14·38	14·57	14·37	14·48	14·60	14·44	14·50	14·97	14·58	14·56	−0·1
15·00	15·38	15·56	15·42	15·47	15·74	15·42	15·21	15·91	15·53	15·53	−0·1
16·00	16·38	16·46	16·58	16·66	16·97	16·60	16·75	…	…	16·67	−0·2

The calculated magnitudes along each horizontal line should show no systematic dependence on the galactic latitude, provided the observed limiting magnitudes have been determined on a uniform photometric scale. This condition is on the whole satisfied. The accidental discrepancies are naturally more pronounced than in the case of van Rhijn's observations, because the star-counts represent here the actually observed values and have not been "smoothed."

The calculated mean magnitudes agree well with the Mount Wilson system in Series I. and II., but are too large by about $0^m{\cdot}2$ in Series III. Apart from this discrepancy, which appears to be of the nature of a correction to the zero-point of the observed magnitudes, the star-counts of Chapman and Melotte confirm the conclusion of the previous section that the limiting photographic magnitudes for any galactic region can be calculated from the observed star counts by means of Tables II. and III.

(*d*) *The Observed Relation between Star-Counts and Visual Magnitudes.*—This relation has been examined in great detail for different galactic zones by Kapteyn in *Groningen Publications*, No. 18, for stars brighter than the 14th magnitude. The limiting magnitudes are based on the Harvard visual system. Some doubt, however, exists regarding the faintest magnitudes which depend on Parkhurst's observations. It is generally recognised that these magnitudes are estimated too bright.

In addition to Kapteyn's work we now possess a valuable series of counts in different galactic latitudes published by Seares in the *Proceedings of the National Academy of Sciences*, 3, No. 3, p. 217. These counts include all the stars down to the limiting magnitude 17·1 on the Harvard visual scale.

A discussion of the material led to the result that the observed relation (m, N) in any latitude can be derived from the relation (m', N') of Table III. if instead of Table II. the following differences ($m' - m$) and log N'/N are substituted.

Table VII.

β.	(Visual Magnitudes.) $(m' - m)$.	$\log N'/N$.
5°	−3·10	−1·99
15	−2·30	−1·47
25	−1·20	−0·82
35	−0·30	−0·31
45	+0·20	−0·03
55	+0·62	+0·19
65	+0·91	+0·35
80	+1·29	+0·54

To verify this conclusion we revert to the observed counts collected in Kapteyn's work in the various tables headed S_1, S_2, etc., and calculate, as before, the corresponding limiting magnitudes

from Tables III. and VII. These calculated magnitudes must agree with those given by Kapteyn in all galactic regions.

In most instances Kapteyn has collected the data into groups comprised between the belts $0° - 20°$, $20° - 40°$, and $40° - 90°$ galactic latitude, and in three cases the star-counts averaged over the whole sky are mentioned. An inspection of Kapteyn's final table shows that the curves (m, N) referring to the four groups are practically identical with the curves for latitudes 10°, 30°, 55°, and 25° respectively.

The limiting magnitudes resulting from these calculations are shown in Table VIII., where the headings of the vertical columns indicate Kapteyn's tables from which the star-counts were taken and the visual limiting magnitudes adopted by him.

TABLE VIII.

β \ m	S_4. 4·50.	S_5. 5·50.	S_4. 6·75.	S_5. 7·62.	S_6. 8·12.	S_8. 8·68.	S_{10}. 9·10.	S_8. 9·25.	S_{11}. 9·96.	S_{11}. 10·86.	S_{13}. 12·14.	S_{14}. 13·18.	S_{13}. 13·90.	S_{11}. 13·90.
5°	...	5·52	...	...	...	...	...	...	...	...	...	...	14·20	...
10	4·54	...	6·74	...	...	...	9·18	9·35	9·84	10·93	12·21	13·39	...	14·24
15	...	5·56	...	...	...	...	...	...	...	...	...	...	14·17	...
25	...	5·56	...	7·58	8·11	8·69	...	...	...	...	...	...	14·25	...
30	4·46	...	6·74	...	...	...	9·18	9·35	9·78	10·87	12·39	13·43	...	14·02
35	...	5·37	...	...	...	...	...	...	...	...	...	...	14·34	...
45	...	5·40	...	...	...	...	...	...	...	...	...	...	14·32	...
55	4·46	5·40	6·71	...	...	...	9·31	9·41	9·82	11·12	12·16	13·37	14·35	14·53
65	...	5·51	...	...	...	...	...	...	...	...	...	...	14·21	...
80	...	5·44	...	...	...	...	...	...	...	...	...	...	14·36	...
Mean	4·49	5·48	6·73	7·58	8·11	8·69	9·22	9·37	9·81	10·97	12·25	13·40	14·27	14·26
Diff.	+·01	+·02	+·02	+·04	+·01	−·01	−·12	−·12	+·15	−·11	−·11	−·22	−·37	−·36

As anticipated, the calculated magnitudes are practically the same for all latitudes. Further, the differences (Kapteyn − Computed) in the last horizontal line show no pronounced systematic progression for magnitudes brighter than the 13th; *i.e.* within this interval, and again at magnitude 17, the calculated magnitudes agree with the Harvard visual system adopted by Kapteyn and Seares. There can be little doubt that the more appreciable differences which are noticed in the faintest groups of Kapteyn's series are due to a systematic error in Parkhurst's magnitudes by which the magnitudes adopted by Kapteyn are influenced.

We have arrived at the significant result that the relation (m, N) in any galactic latitude, whether referring to photographic or visual magnitudes, can be derived from one and the same relation (m', N′) shown in Table III. simply by converting m' and log N′ into m and log N by means of the corrections given in Tables II. and VII. respectively.

Tables II., III., VII. are therefore sufficient to calculate the relation (m, N) for any given galactic latitude. Table A, showing

the limiting magnitudes m_v (Harvard visual) and m_p (Mount Wilson photographic) corresponding with specified values of log N for galactic latitudes 0°, 10° . . . 90° has been calculated in this manner.

It shows clearly that the same star number, N, corresponds with essentially different limiting magnitudes according to whether these have been standardised on the visual or the photographic system. The differences $(m_p - m_v)$ are rapidly increasing with apparent magnitude, the rate of increase being greatest in the galactic plane and least at the poles.

The reality of this conspicuous difference between visual and photographic magnitudes is well illustrated in the case of the Mount Wilson star-counts. According to Seares, these counts include all stars whose *visual* magnitudes are brighter than 17·1. The last column of Table VIII. shows how well they satisfy the conditions of Tables III. and VII. But if we calculate with Seares's star-counts the photographic magnitudes from Tables III. and II. we get the following result (third column):—

β.	log N.	m_p.	log N $m_p = 17{\cdot}7$.
5°	4·12	19·8	3·60
15	3·86	19·3	3·48
25	3·53	18·8	3·26
35	3·26	18·7	3·06
45	3·10	18·7	2·89
55	2·98	18·7	2·78
65	2·89	18·6	2·70
80	2·79	18·6	2·62

Seares assumed a colour-index of +0·6, which would make the limiting photographic magnitude 17·7. The star-counts calculated for this magnitude from Tables II. and III. are shown in the last column. The figures show clearly that the observed star-counts cannot be reconciled with the adopted magnitude 17·7, if van Rhijn's figures, on which Table II. and III. are based, are accepted.

That indeed a much larger colour-correction is required, is made evident from the direct comparison of photographic and photo-visual magnitudes of the north-polar sequence published by Seares in *Contributions of the Mount Wilson Observatory*, No. 97. Table IX. gives the average colour-indices compiled from Seares's Catalogue for specified values of m_p.

TABLE IX.

Colour-Indices of Stars of the North-polar Sequence (Seares).

m_p.	$(m_p - m_v)$.	No. of Stars.	m_p.	$(m_p - m_v)$.	No. of Stars.	m_p.	$(m_p - m_v)$.	No. of Stars.
5·88	+0·25	11	14·59	+0·86	25	17·59	+1·08	64
10·25	+0·66	21	15·43	+1·01	29	18·25	+1·35	38
13·36	+0·87	17	16·52	+0·93	40	18·72	+1·57	54
						19·09	+1·79	12

The galactic latitude of the North Pole is 27°. For this latitude Seares's star-counts show $m_p = 18{\cdot}7$ corresponding with $m_v = 17{\cdot}1$; *i.e.* a colour-index of $+1{\cdot}6$, which agrees almost exactly with that found in the north-polar sequence (Table IX.).

The photometrically determined colour-indices of Table IX. show, in accordance with Table A, a pronounced increase in the colour-index with apparent magnitude. This result is confirmed for the brighter stars by photometric investigations of the colour-indices of the stars in the C.P.D. By comparing the C.P.D. system with the Harvard visual system on the one hand and the Cape photographic system on the other, the following colour-indices (Cape ptg. – Harv. vis.) were obtained in order of apparent photographic magnitude.

Mean Galactic Latitude, $\beta = 25°$.

m_p.	Colour-Index. C.P.D.	Table A.	Table IX.
7·4	+0·46	+0·46	+0·40
8·2	+0·48	+0·50	+0·46
9·0	+0·52	+0·53	+0·52
9·9	+0·60	+0·60	+0·60
10·7	+0·68	+0·67	+0·67
11·5	+0·73	+0·73	+0·73

Here again a distinct increase with apparent magnitude is noticed, in good agreement with Tables A and IX.

These observations point to the conclusion that Table A represents the relation between star numbers and limiting magnitudes (visual or photographic), and hence that Tables III., II., and VII. are sufficient to establish these relations for all galactic latitudes, with this restriction, however, that they refer only to the *average* conditions prevailing along the particular galactic parallel selected. It has thus been demonstrated that the observed relations (m, log N), for both photographic and visual magnitudes, satisfy equation (12).

§ 3. *The Observed Relation between Apparent Magnitudes and Mean Parallaxes.*

The investigations of the parallactic motions of the stars with known transverse proper motions have furnished a statistical relationship between the apparent magnitudes, m, and the mean parallaxes of stars of that magnitude, π_m.

From a discussion of the transverse motions of the stars in the Bradley-Auwers Catalogue, Kapteyn found that, averaged over the whole sky, this relation is approximately represented by an equation of the form

$$\log \pi_m = -P_0 - P_1 m \quad . \quad . \quad . \quad (15)$$

where P_0 and P_1 represent constants. More recent investigations by Dyson and Thackeray have confirmed the general form of this

TABLE A.

log N.	$\delta=0°$.		10°.		20°.		30°.		40°.		50°.		60°.		70°.		80°.		90°.	
	m_v.	m_p.	m_v.	m_p.	m_v.	m_p.	m_v.	m_p.	m_v.	m_p.	m_v.	m_p.	m_v.	m_p.	m_v.	m_p.	m_v.	m_p.	m_v.	m_p.
−1·60	...	...	...	...	...	...	...	...	4·87	5·30	4·90	5·43	4·93	5·50	4·92	5·53	4·95	5·56	4·96	5·57
−1·40	...	...	...	...	5·02	5·44	5·18	5·60	5·27	5·71	5·30	5·84	5·34	5·91	5·32	5·94	5·36	5·97	5·37	5·98
−1·20	5·08	5·70	5·22	5·72	5·41	5·84	5·57	6·01	5·67	6·12	5·71	6·25	5·74	6·32	5·73	6·35	5·77	6·39	5·78	6·40
−1·00	5·47	6·09	5·61	6·12	5·80	6·24	5·97	6·41	6·08	6·53	6·12	6·66	6·15	6·74	6·14	6·77	6·18	6·81	6·20	6·82
−·80	5·86	6·49	6·00	6·52	6·20	6·65	6·37	6·82	6·48	6·95	6·53	7·08	6·56	7·16	6·56	7·20	6·60	7·24	6·62	7·26
−·60	6·25	6·89	6·39	6·92	6·59	7·05	6·78	7·23	6·89	7·36	6·94	7·51	6·98	7·60	6·99	7·64	7·03	7·70	7·05	7·71
−·40	6·64	7·29	6·79	7·33	6·99	7·46	7·18	7·65	7·30	7·79	7·36	7·95	7·40	8·05	7·42	8·10	7·48	8·16	7·49	8·17
−·20	7·03	7·70	7·18	7·74	7·39	7·87	7·59	8·08	7·72	8·23	7·78	8·40	7·84	8·51	7·86	8·57	7·93	8·62	7·94	8·65
·00	7·42	8·11	7·57	8·14	7·79	8·29	8·00	8·51	8·14	8·67	8·20	8·85	8·29	8·99	8·31	9·05	8·38	9·10	8·41	9·13
+·20	7·81	8·52	7·97	8·56	8·20	8·71	8·42	8·95	8·57	9·14	8·66	9·33	8·75	9·47	8·78	9·54	8·86	9·60	8·90	9·63
+·40	8·21	8·94	8·37	8·98	8·61	9·15	8·84	9·40	9·03	9·61	9·13	9·82	9·24	9·97	9·26	10·05	9·35	10·11	9·40	10·14
+·60	8·61	9·36	8·78	9·40	9·02	9·60	9·28	9·87	9·49	10·10	9·61	10·33	9·72	10·49	9·75	10·58	9·86	10·65	9·91	10·68
+·80	9·01	9·79	9·18	9·83	9·44	10·06	9·73	10·36	9·96	10·61	10·09	10·84	10·22	11·03	10·27	11·12	10·38	11·20	10·43	11·23
+1·00	9·42	10·24	9·59	10·28	9·86	10·52	10·19	10·86	10·43	11·13	10·59	11·38	10·83	11·58	10·80	11·68	10·92	11·77	10·98	11·80
+1·20	9·83	10·69	10·00	10·75	10·30	11·00	10·67	11·37	10·93	11·67	11·11	11·95	11·27	12·15	11·36	12·27	11·49	12·38	11·56	12·41
+1·40	10·24	11·16	10·42	11·22	10·75	11·49	11·15	11·90	11·45	12·24	11·64	12·53	11·82	12·76	11·94	12·89	12·08	13·01	12·15	13·04
+1·60	10·65	11·65	10·85	11·71	11·20	12·01	11·65	12·46	11·99	12·83	12·20	13·13	12·40	13·39	12·53	13·54	12·70	13·68	12·77	13·71
+1·80	11·07	12·14	11·28	12·22	11·68	12·55	12·16	13·04	12·54	13·45	12·78	13·78	13·01	14·06	13·15	14·23	13·33	14·37	13·42	14·41
+2·00	11·50	12·65	11·72	12·74	12·16	13·09	12·70	13·63	13·11	14·08	13·39	14·45	13·65	14·76	13·81	14·95	14·01	15·10	14·11	15·15
+2·20	11·94	13·18	12·17	13·28	12·66	13·67	13·25	14·26	13·72	14·76	14·03	15·16	14·32	15·51	14·50	15·72	14·74	15·88	14·84	15·94
+2·40	12·39	13·73	12·64	13·84	13·17	14·27	13·82	14·92	14·35	15·48	14·69	15·91	15·02	16·30	15·24	16·53	15·49	16·73	15·62	16·78
+2·60	12·85	14·32	13·12	14·42	13·71	14·89	14·43	15·61	15·02	16·23	15·39	16·71	15·76	17·14	16·01	17·41	16·30	17·63	16·44	17·69
+2·80	13·32	14·91	13·61	15·03	14·26	15·56	15·06	16·35	15·71	17·03	16·15	17·57	16·56	18·06	16·84	18·35	17·16	18·59	17·32	18·66
+3·00	13·82	15·53	14·13	15·68	14·84	16·26	15·73	17·12	16·45	17·90	16·95	18·48	17·40	19·03	17·73	19·38				
+3·20	14·33	16·18	14·66	16·35	15·44	16·98	16·43	17·95	17·23	18·82										
+3·40	14·84	16·88	15·21	17·05	16·07	17·75	17·17	18·85												
+3·60	15·38	17·62	15·79	17·81	16·74	18·60														
+3·80	15·95	18·39	16·38	18·61	17·43	19·49														
+4·00	16·54	19·22	17·00	19·47																
+4·20	17·16	20·12																		

equation, with only slight modifications of the constant factors. Still later results, discussed at the Groningen Astronomical Laboratory, have been provisionally published by Schouten in his paper "On the Determination of the Principal Laws of Statistical Astronomy" (p. 41). These results are particularly important for our purpose, as for the first time the material has been examined in relation to galactic latitude, and therefore permits an examination of the relation (m, π_m) in different galactic regions.

For five galactic zones Schouten gives the following equations, which satisfy the observed data:—

$$\left.\begin{array}{rrrr} \beta = & 5^\circ & \log \pi_m = & -1{\cdot}117 \quad -0{\cdot}142\ m \\ & 20 & & -1{\cdot}096 \quad -0{\cdot}142\ m \\ & 38 & & -1{\cdot}043 \quad -0{\cdot}142\ m \\ & 58 & & 0{\cdot}976 \quad -0{\cdot}142\ m \\ & 75 & & -0{\cdot}934 \quad -0{\cdot}142\ m \end{array}\right\} \quad . \quad . \quad (16)$$

the magnitudes being based on the Harvard visual system. The investigation extends over stars brighter than about 10·5. Equations (16) show at once that by substituting

$$\log \pi'_{m'} = \log \pi_m + \Delta_\pi\,;\ m' = m + \Delta_m,$$

where Δ_π and Δ_m are numerical corrections depending on β, the equations (16) may be reduced to a standard form:

$$\log \pi'_{m'} = -\mathrm{P}'_0 - 0{\cdot}142\ m'.$$

Hence the observations satisfy formally the conditions of equation (13).

The corrections Δ_π and Δ_m cannot, however, be arbitrarily chosen. For F to be of the form (7) it is necessary that $\Delta_m = m' - m$, the latter quantities being in numerical agreement with those of Table VII. Further, we have to adopt $\Delta_\pi = -\log \alpha/\alpha' = -\frac{1}{3}\log \mathrm{N}'/\mathrm{N}$, where $\log \mathrm{N}'/\mathrm{N}$ is represented by the corresponding quantities of the same table. Taking these conditions into account, the relation (π_m, m) of any cone is represented by the equation:

$$\log \pi_{m'} = -\mathrm{P}'_0 + [\tfrac{1}{3}\log \mathrm{N}'/\mathrm{N} - 0{\cdot}142(m' - m)] - 0{\cdot}142\ m \quad (17)$$

where P'_0 represents a universal constant and the bracketed term is calculated with the quantities given in Table VII. for the galactic latitude selected.

Equation (17) is in satisfactory agreement with the observations—which in their present state cannot lay claim to very great accuracy—if we adopt $\mathrm{P}'_0 = 0{\cdot}945$. Thus, for instance, calculating π_m for $m = 7{\cdot}0$, and for the above five galactic zones from Schouten's equations (16) on the one hand, and from equation (17) and Table VII. on the other, we find:

	π_7.	
β.	Schouten.	Equation (17).
5°	″0·0077	″0·0069
20	0·0080	0·0083
38	0·0092	0·0100
58	0·0107	0·0108
75	0·0118	0·0112

Combining the result derived from the parallaxes with that obtained in the previous section from the star-counts, we arrive at the conclusion that the distribution-function F is of such a character that the conditional equations (12) and (13) are satisfied. Hence it has the form (7), viz.

$$F[ar, h(M + A)],$$

where h has the same value for all cones, while a and A depend on the spherical co-ordinates (λ, β).

If a' and A′ refer to a selected standard cone, viz. that in which the relation (m', log N′) is represented by Table III., we have

$$\log a/a' = \tfrac{1}{5} \log N'/N\,;\ A - A' = m' - m - \tfrac{2}{5} \log N'/N,$$

where the quantities log N′/N and $(m' - m)$ are obtained from Table II. for photographic magnitudes, and from Table VII. for visual magnitudes. Table X. shows the values of a/a' and A − A′ thus calculated.

TABLE X.

Photogr. Magnitudes.			Visual Magnitudes.		
β.	a/a'.	A − A′.	β.	a/a'.	A − A′.
0°	0·55	0·00	5°	0·22	+0·22
10	0·59	0·00	15	0·32	+0·25
20	0·75	−0·01	25	0·53	+0·17
30	1·02	−0·04	35	0·79	+0·22
40	1·29	−0·04	45	0·98	+0·25
50	1·48	−0·10	55	1·16	+0·30
60	1·67	−0·12	65	1·31	+0·33
70	1·81	−0·11	80	1·51	+0·39
80	1·91	−0·11			
90	1·94	−0·11			

The figures show two prominent features. First, the extremely small change of A with galactic latitude will be noticed in both instances. The variations, though systematic, are indeed so insignificant, that in consideration of the probable uncertainties in the determinations of the zero-point of the magnitude scales which would enter into the quantities A with their full amount,

their reality may well be doubted. On the other hand, the variations of α with latitude are very considerable, and it will be noticed that throughout those determined from the visual magnitudes are appreciably smaller than those derived from the photographic magnitudes. This feature will be of special importance in the subsequent discussion.

The results of the preceding investigation can thus be summarised in the statement that the distribution-function of the stellar system surrounding the sun is represented in the form

$$\mathrm{F}[ar,\ h(\mathrm{M}+\mathrm{A})],$$

where h is a universal constant, A depends on the effective wave-length but is independent of the positional co-ordinates, and a is a function of the effective wave-length and of the spherical co-ordinates λ, β.

Having found the relation between the distribution-functions of different cones, the next step is to establish an analytical form of F which satisfies the statistical data observed in the standard cone.

§ 4. *The Analytical Form of the Distribution-Function F.*

The theories advanced by Seeliger, Kapteyn, Schwarzschild and others regarding the analytical character of the distribution-function are based on the assumption that F can be expressed in the form of a product

$$\mathrm{F}=f(ar)\psi[h(\mathrm{M}+\mathrm{A})] \quad . \quad . \quad . \quad (18)$$

where f is a function of r, λ, β, but independent of M, whereas ψ is a function of M and independent of r, λ, β.

Defining ψ in such a manner that

$$\int_{-\infty}^{+\infty}\psi\, d\mathrm{M}=1,$$

we have

$$\int_{-\infty}^{+\infty}\mathrm{F}\, d\mathrm{M}=f(ar),$$

this expression denoting the total number of stars in a cube whose side is 1 parsec. situated at distance r. The function f represents, therefore, the law according to which the star-*density* changes with distance. The function ψ, on the other hand, denotes the law according to which the absolute magnitudes are distributed among the individual stars within that volume.

In the theories mentioned this "luminosity"-law ψ is supposed to be the same at all distances.

With the statistical data at our disposal the function f can be evaluated. Schouten, in the memoir quoted, gives a summary of the results obtained by Kapteyn, Schwarzschild, Seeliger, and himself. Taking the mean of all determinations, which are fairly accordant at large distances but differ considerably in the vicinity

of the sun, the following values of f at specified distances were obtained, half weight being given to the determinations marked uncertain in Schouten's summary.

TABLE XI.

Limiting log r.	$f(ar)$.
3·00	
	0·186
2·80	
	0·274
2·60	
	0·426
2·40	
	0·601
2·20	
	0·741
2·00	
	0·853
1·80	
	0·941
1·60	
	1·000
1·40	
	1·06
1·20	
	1·05
1·00	
	1·07
0·80	
	1·09
0·60	

According to these theories the total number of stars in unit volume decreases with increasing distance.

The luminosity-law, ψ, has been found to be represented in the form of an error law:

$$\psi = \frac{h}{\sqrt{\pi}} e^{-h^2(\mathrm{M}+\mathrm{A})^2} = \frac{h}{\sqrt{\pi}} e^{-h^2[m-5\log r+\mathrm{A}]^2} \quad . \quad . \quad (19)$$

With proper choice of the constants h, A, and a, the functions f and ψ can be made to represent the relation (m, log N) shown in Table III. and the relation (m, log π_m) expressed in equation (17), and therefore to satisfy in these two respects the observational data, at least within the errors of the latter.

The question, however, arises whether these are the *only* forms of f and ψ by which the statistical data are satisfied. Naturally, the assumed analytical forms of these functions can only be accepted if they satisfy the observed conditions *uniquely*. If, on the other hand, it can be shown that different forms of f and ψ represent the two relations (m, log N) and (m, log π_m) within the errors of observation, it becomes necessary to establish from observations a third criterion by which it can be decided which of these optional forms has to be chosen.

Let an alternative form of the function F be introduced in the following manner.

The number of stars between $(r \pm \frac{1}{2}dr)$ and $(m \pm \frac{1}{2}dm)$ is given by

$$dN = f\,\psi\, r^2\, dr\, dm\, d\omega \quad . \quad . \quad . \quad (20)$$

Let us define a variable ρ as a function of r, viz.

$$r = g(\rho) \quad . \quad . \quad . \quad . \quad . \quad (21)$$

and let $\phi(\rho)$ be a function which satisfies the condition

$$fr^2\,dr = \phi\,\rho^2\,d\rho \quad . \quad . \quad . \quad . \quad (22)$$

The absolute magnitude M is therefore expressed by

$$\mathrm{M} = m - 5\log g(\rho) \quad . \quad . \quad . \quad (23)$$

and

$$\left.\begin{aligned} d\mathrm{N} &= \phi\,\psi\,\rho^2\,d\rho\,dm\,d\omega, \\ \psi &= \frac{h}{\sqrt{\pi}} e^{-h^2[m-5\log g(\rho)+\mathrm{A}^2]} \end{aligned}\right\} \quad . \quad . \quad . \quad (24)$$

Let us now make a special assumption regarding $g(\rho)$ and write

$$r = \rho\,.\,e^{\beta\rho}\,;\ \beta = 0{\cdot}000949 \quad . \quad . \quad . \quad (25)$$

Computing the values of ρ corresponding with the limiting values r of the spherical shells in Table XI., it is found that equation (22) is satisfied if a constant value of $\phi = 1{\cdot}14$ is assumed for all shells. The calculated numerical values of $fr^2\,dr$ and $\phi\,\rho^2\,d\rho$ are shown in Table XII.

TABLE XII.

$\log r$.	$\log \rho$.	$\log [fr^2 dr]$.	$\log [\phi\rho^2 d\rho]$.
3·000	2·7619		
2·800	2·6259	8·745	8·748
2·600	2·4765	8·334	8·364
2·400	2·3149	7·925	7·935
2·200	2·1428	7·475	7·465
2·000	1·9622	6·966	6·959
1·800	1·7754	6·427	6·425
1·600	1·5842	5·870	5·869
1·400	1·3899	5·296	5·299
1·200	1·1936	4·721	4·718
1·000	0·9959	4·117	4·131
0·800	0·7954	3·525	3·539
0·600	0·5984	2·935	2·944

Equations (24) take in this case the form

$$\left.\begin{aligned} d\mathrm{N} &= \phi\,.\,\psi\,.\,\rho^2\,d\rho\,dm\,d\omega \\ \psi &= \frac{h}{\sqrt{\pi}} e^{-h^2[m-5\log\rho-0{\cdot}000\,946\rho+\mathrm{A}]^2} \\ \phi &= \text{constant} = 1{\cdot}14 \end{aligned}\right\} \quad . \quad . \quad (26)$$

whereas the original equations expressed in terms of r are

$$\left.\begin{aligned} dN &= f.\psi\, r^2\, dr\, dm\, d\omega \\ \psi &= \frac{h}{\sqrt{\pi}} e^{-h^2[m-5\log r+A]^2} \\ f &= \text{function of } r \text{ (Table XI.)} \end{aligned}\right\} \quad . \quad . \quad (27)$$

The number of stars brighter than m is obtained by integrating with regard to m from $-\infty$ to m and with regard to either ρ or r from 0 to ∞. The resulting values of N_m must be the same in both cases.

So far as the relation (m, log N) is concerned, it is, therefore, irrelevant whether ρ or r are accepted as the true distances, and hence it is likewise irrelevant whether the density- and luminosity-laws are accepted in accordance with equations (26) or (27).

Let us next examine the parallaxes. Expressed in seconds of arc, the parallax would be in the first case $1/\rho$, in the second case $1/r$. But according to (25)

$$1/r = 1/\rho e^{-\beta\rho} = 1/\rho - 1/\rho[1 - e^{-\beta\rho}].$$

Since $[1 - e^{-\beta\rho}] < \beta\rho$, we find that the difference

$$1/\rho - 1/r < \beta, \quad \text{or} \quad < 0{\cdot}0009.$$

Since this applies to all parallaxes, we conclude that the mean parallaxes of stars of a given magnitude m computed with the density- and luminosity-laws of equations (26) and (27) differ by less than $\frac{1}{1000}$ of a second of arc.

It will be readily admitted that the question which of the two assumptions represents the *true* parallaxes cannot be settled by our present knowledge of the mean parallaxes derived from the transverse proper motions.

Hence the conclusion is reached, that neither the observed relation (m, log N) nor the relation (m, log π_m) can decide which of the two alternative forms of the density- and luminosity-law has to be adopted.

In view of this difficulty, it is necessary to search for other criteria. One of these, pointing against the acceptance of the frequency function F in the form (18), has already been mentioned. The parameter a was found to depend largely on the effective wave-length for which the magnitudes m were determined. It is inconceivable that the total number of stars within unit volume could be a function of any purely *optical* characteristic. Hence, the assumption involved in equation (18) that a should be *exclusively* associated with the f-function cannot be maintained. It is necessary that a, which is combined with the distance in the form of a product $a\rho$ or ar, should appear in that part of the distribution-function, viz. ψ, which depends on the *optical* conditions of the stars. It may, of course, without further evidence,

appear in both f and ψ, *i.e.* the assumption expressed in equations (26) represents only one of an infinity of possible cases.

It is unfortunate that the parallactic motions supply the average *parallaxes*, and not the average *distances*. If the latter could be obtained from this source, it is evident from the form of equation (25) that the differences between r_m and ρ_m would be appreciable even at comparatively short distances, and the question as to which of the two assumptions represents the true distances could probably be decided.

Evidently we have to search for observational material which independently fixes the true distances and which may be compared with similar data derived from the two above assumptions.

Such material is supplied in the researches of Dyson and Eddington concerning the distribution in distance of the stars contained in the Carrington Circumpolar Catalogue and in Boss's General Catalogue, both of which may be regarded as *complete* collections of stars brighter than a fixed limiting magnitude. Considering that the bulk of the stars possess magnitudes near the limiting magnitude of the catalogue it is legitimate to consider Dyson's and Eddington's results as representative of the distribution in distance of stars of a given apparent magnitude.

Now, according to the first of equations (27), the number of stars of a given apparent magnitude between $(m \pm \frac{1}{2})$ and lying between the distances $(r \pm \frac{1}{2}\,dr)$ is

$$dN = f\,\psi_m\,r^2\,dr\,d\omega \quad . \quad . \quad . \quad (28)$$

Corresponding numbers are found from the first of equations (26) for stars between $(\rho \pm \frac{1}{2}d\rho)$, if $r = \rho e^{\beta\rho}$, viz.

$$dN = \phi\,\psi_m\,\rho^2\,d\rho\,d\omega \quad . \quad . \quad . \quad (29)$$

Comparing r and ρ corresponding with specified numbers with the distances derived by Dyson and Eddington for the same numbers, we are in a position to decide whether r, ρ, or any intermediate value between r and ρ correspond with the distances derived by these investigators.

The analytical forms of f and ψ which are required for the evaluation of dN in (28) can be derived from the mathematical theory of Schwarzschild given in his paper "Zur Stellarstatistik" in *A.N.*, 4557.

Schwarzschild showed that if the numbers of stars of given magnitude, b_m, can be expressed by an equation of the form

$$\log b_m = a_0 - a_1 m - a_2 m^2 \quad . \quad . \quad . \quad (30)$$

and the mean parallaxes π_m by an equation of the form

$$\log \pi_m = \mathrm{P}_0 - \mathrm{P}_1 m \quad . \quad . \quad . \quad . \quad (31)$$

the function f has the form

$$\log f = a_0 + 5a_1 \log r - 25a_2(\log r)^2 \quad . \quad . \quad (32)$$

while the function ψ is expressed by

$$\log \psi = b_0 - b_1 M - b_2 M^2 \quad . \quad . \quad . \quad (33)$$

the quantities a and b being constants within the cone $d\omega$.

Now the relation between m' and $b'_{m'}$ (which is equivalent with $\frac{dN'_m}{dm'}$), can be found from the figures of Table III. Excepting the magnitudes brighter than 4, the relation $(b'_{m'}, m')$ is satisfactorily represented by the equation

$$\log b'_{m'} = -4{\cdot}465 + 0{\cdot}638m' - 0{\cdot}0125m'^2 \quad . \quad . \quad (34)$$

which is in formal agreement with (30)

We have also seen that the parallaxes π'_m and m' are connected by the equation

$$\log \pi'_{m'} = -0{\cdot}945 - 0{\cdot}142m' \quad . \quad . \quad . \quad (35)$$

which formally agrees with (31).

Hence the functions f and ψ must be, at least approximately, of the form (32) and (33), except in the immediate neighbourhood of the sun, where equation (30) breaks down.

In the case of visual magnitudes with which we are here concerned, Table VII. enables us to find equations corresponding with (34) and (35) for any galactic latitude. Adopting a mean latitude 25°, we find

$$\left.\begin{aligned} \log b_m &= -4{\cdot}429 + 0{\cdot}668m - 0{\cdot}0125m^2 \\ \log \pi_m &= -1{\cdot}048 - 0{\cdot}142m. \end{aligned}\right\} \quad . \quad . \quad (36)$$

From the coefficients of these two equations the quantities $(a_0 + b_0)$, a_1, a_2, b_1, b_2 may then be computed. The first of these is irrelevant, however, if the numbers are expressed in percentages. Schwarzschild's theory gives the following numerical values:

$$a_1 = +0{\cdot}281 \; ; \; b_1 = -0{\cdot}546$$

$$a_2 = +0{\cdot}0176 \; ; \; b_2 = +0{\cdot}0430.$$

It is easily seen that the number of stars of given apparent magnitude $(m \pm \frac{1}{2})$ lying between distances $(r \pm \frac{1}{2}dr)$ and expressed in percentages of the total number between $r = 0$ and $r = \infty$ is given by the formula

$$dN = 105{\cdot}4 \quad 10^{-1{\cdot}515[\log r - 1{\cdot}213 - 0{\cdot}142m]^2} d(\log r) \quad . \quad (37)$$

from which the number contained within any sphere of radius r can be calculated. Table XIII. shows the percentages of stars of magnitudes $6^{m}{\cdot}2$ and $10^{m}{\cdot}2$ included within spheres whose radii r are given.

TABLE XIII.

Percentages.	log r. m 6·2.	log r. m 10·2.
0·3	1·060	1·628
1·4	1·260	1·828
4·7	1·460	2·028
12·6	1·660	2·228
26·9	1·860	2·428
46·5	2·060	2·628
67·0	2·260	2·828
83·4	2·460	3·028
93·3	2·660	3·228
97·9	2·860	3·428
99·5	3·060	3·628

The magnitudes 6·2 and 10·2 have been selected because they agree approximately with the limiting magnitudes of the Boss and Carrington Catalogues.

Further, the luminosity-law $\psi(M)$ is expressed by

$$\left.\begin{aligned} \psi(M)dM &= 0{\cdot}1775 \quad 10^{-0{\cdot}043[M-1{\cdot}7]^2}dM \\ \int_{-\infty}^{+\infty}\psi(M)dM &= 1 \\ M &= m - 5\log r \end{aligned}\right\} \quad . \quad (38)$$

Evidently, if the distances r represent the true distances, the distributions shown in Table XIII. should agree with those found by Eddington and Dyson from the p.m., provided that their distributions represent the true conditions.

Let us now substitute ρ in place of r. From (25) we have the relation

$$\log r = \log\rho + \beta\log e\,.\,\rho = \log\rho + 0{\cdot}000412\rho \quad . \quad (39)$$

The density-law is now

$$\phi = \text{constant},$$

and the luminosity-law is expressed by

$$\psi(M) = 0{\cdot}1775 \quad 10^{-0{\cdot}043[m-5\log\rho-0{\cdot}00206\rho-1{\cdot}7]} \quad . \quad (40)$$

Hence the number of stars of magnitude m in unit field lying between $(\rho \pm \frac{1}{2}d\rho)$ is

$$dN = \text{constant} \times 10^{-0{\cdot}043[m-5\log\rho-0{\cdot}00206\rho-1{\cdot}7]^2}\rho^2 d\rho \quad . \quad (41)$$

the constant being determined by the condition that the total number is 100. The resulting distributions for stars of magnitudes $6^{m}{\cdot}2$ and $10^{m}{\cdot}2$ are shown in Table XIV., which gives the percentages contained within spheres of specified radii ρ centred in the sun.

TABLE XIV.

$m=6\cdot2$.		$m=10\cdot2$.	
log ρ.	Percentages.	log ρ.	Percentages.
1·000	0·4	1·800	1·0
1·200	1·4	2·000	4·3
1·400	4·3	2·200	13·4
1·600	11·1	2·400	32·4
1·800	24·3	2·600	60·7
2·000	44·2	2·800	86·7
2·200	67·1	3·000	98·4
2·400	86·0		
2·600	96·8		
2·800	100		

The distributions of Tables XIII. and XIV. may now be compared with those of Eddington and Dyson. This is done graphi-

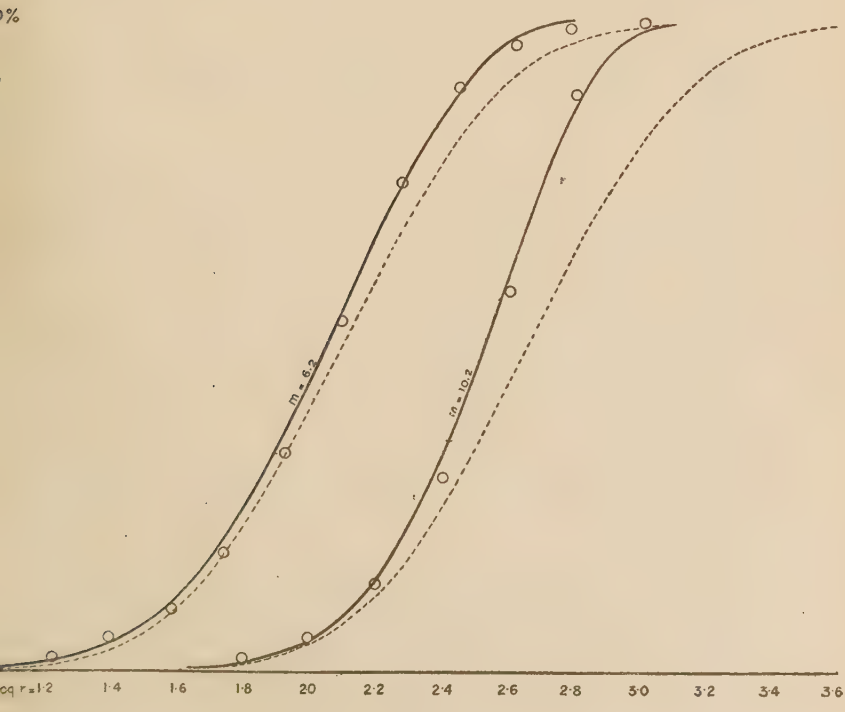


FIG. 1.

cally in fig. 1. The abscissæ represent the logs of the limiting radii r and ρ respectively, the ordinates denote the corresponding percentages.

The dotted curves show the distributions according to Table XIII., the drawn-out curves those of Table XIV. Lastly, the open circles represent points of the distribution curves derived from the p.m. by Eddington and Dyson.

It is seen at once that these latter curves are in good agreement with those of Table XIV., but differ essentially from those of Table XIII. Hence, so far as the evidence from Eddington's and Dyson's investigations is entitled to weight, the true distances are represented by ρ and not by r.

Of the two alternative assumptions regarding the functions f and ψ, viz.

$$\left.\begin{array}{l|l} f = 10^{a_0 + 5a_1 \log r - 25a_2 (\log r)^2}; & f = \text{constant} \\ \psi = 10^{b_0 - b_1 M - b_2 M^2} = \dfrac{h}{\sqrt{\pi}} e^{-h^2[M - M_0]^2}; & \psi = \dfrac{h}{\sqrt{\pi}} e^{-h^2[M - a\rho - M_0]^2} \\ M = m - 5 \log r; & M = m - 5 \log \rho; \; a = 0{\cdot}00206 \end{array}\right\} \quad (42)$$

$$h = 0{\cdot}315; \; M_0 = 1{\cdot}7,$$

the latter satisfies the observational data far better than the former. There are, thus, two reasons for the adoption of the second hypothesis and for the rejection of the first. First, it interprets a as a quantity depending on *optical* causes and thereby explains the dependence of a on the effective wave-lengths for which the magnitudes were determined. Secondly, it satisfies the distribution in distance of stars of given magnitude derived by Eddington and Dyson. In both respects the first hypothesis does not meet the requirements.

The adoption of the second hypothesis would necessitate a fundamental change of our conceptions regarding the structure of the universe. Hitherto, guided by the results obtained from the first hypothesis, the conclusion was drawn that the stellar system surrounding the sun represents a system in itself, separated from other similar systems by interstices of space void of star matter. The conception of an "island" universe is based on the alleged fact that the frequency of star matter, f, decreases with increasing distance. Accordingly, the apparent condensation in the galaxy is due to a denser crowding of stars in the neighbourhood of the galactic plane.

The second hypothesis leads to a radically different conception. Here, the frequency, f, is constant at all distances, *i.e.* the idea of an "island" universe is abandoned. Star matter is *evenly* distributed at least as far as the outmost confines of that portion of the universe which is accessible to observation. The apparent heterogeneity of distribution, as evidenced in the phenomenon of the galaxy, is purely optical, the intrinsic luminosities of the galactic stars being greater than those of stars in other regions.

Following this hypothesis to its logical conclusions, we have to draw the inference that wherever an apparent condensation of stars presents itself to our eyes, whether in the galaxy or in star

clusters or the Magellanic clouds, the physical cause of such crowding is *optical*, *i.e.* due to a departure of the luminosity-law from its normal condition.

Let us consider the special case of the so-called globular clusters. The idea commonly held is that they form isolated dynamical systems in which the frequency of stars is a function of distance from the centre. Recent investigations have revealed a quasi-elliptical form of their projections on the sky, and from this fact the inference was drawn that clusters represent "island" universes similar to the one surrounding the sun, with a galactic condensation similar to ours and a gradual thinning out of star matter with increasing distance from the centre of gravity. Assuming, for the moment, this analogy to be justified, let us examine whether the hypothesis of a *uniform* luminosity-law, on which the older conclusions regarding the constitution of our own system as an island are essentially based, applies to clusters. In other words, we shall inquire whether the assumption that the frequency, f, is a function of the distance, d, from the centre of the cluster, while the luminosity-law, ψ, is the same at all distances (first hypothesis), is in agreement with observed facts, or whether it is necessary to interpret the luminosity-law as a function of d (second hypothesis).

§ 5. *The Distribution of Stars in Clusters.*

The researches of Shapley and others have demonstrated conclusively that the distances of globular clusters from the sun are great in comparison with their linear dimensions. This fact may be utilised with advantage in the definition of the luminosity-law ψ, since we may assume, with sufficient approximation, $M = m - 5 \log r_0$, where r_0 is the distance of the centre of the cluster from the sun. Hence the luminosity-law, in its most general form, and in accordance with the first hypothesis is

$$\psi[h(M - M_0)] = \psi[h(m - m_0)]$$
$$m_0 = M_0 + 5 \log r_0,$$

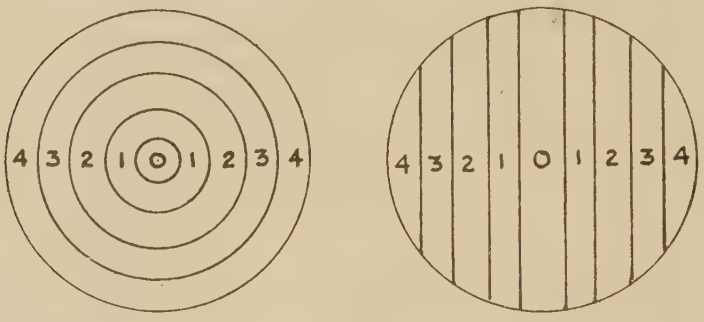
h and m_0 being independent of the distance d. We may thus operate with the apparent magnitudes in lieu of the absolute magnitudes.

Let, in fig. 2, A represent the projection of the cluster on the sky and let the outer circle be the effective boundary beyond which the number of cluster stars may be considered negligible. Let the total area of the projection be subdivided into a central circular area and concentric rings, 0, 1, 2, . . . etc. We assume that two photographs of the cluster have been taken with widely different times of exposure, t_a and t_b, and that on each photograph all the stars visible within the rings have been counted. Let the limiting magnitudes (*i.e.* magnitudes of the faintest stars included in the counts) be m_a and m_b respectively.

Further, let B represent a section containing the line of sight.

(The cluster is assumed spherical, but it will be noticed that its actual shape is irrelevant.)

The effective volume of a ring is represented by its area (A) multiplied into the length of the corresponding median parallel



A. FIG. 2. B.

line (B). If these volumina are denoted by v_0, v_1, . . ., and the mean star frequences by f_0, f_1, . . ., the *total* numbers of stars contained in the cylindrical sections are

$$f_0 v_0, f_1 v_1 \ . \ . \ .$$

Hence, on the photograph of exposure t_a, the numbers counted within each ring are represented by

$$n_{0,a} = f_0 v_0 \int_{-\infty}^{m_a} \psi[h(m-m_0)]\,dm\ ;\ n_{1,a} = f_1 a_1 \int_{-\infty}^{m_a} \psi[h(m-m_0)]\,dm, \text{etc.},$$

while the corresponding numbers on the photograph of exposure t_b are

$$n_{0,b} = f_0 v_0 \int_{-\infty}^{m_b} \psi[h(m-m_0)]\,dm\ ;\ n_{1,b} = f_1 a_1 \int_{-\infty}^{m_b} \psi[h(m-m_0)]\,dm, \text{ etc.}$$

Hence, the ratios of the counts within the same ring are

$$\frac{n_{0,b}}{n} = \frac{n_{1,b}}{n_{1,a}} = \ . \ . \ . = \frac{\int_{-\infty}^{m_b} \psi\,dm}{\int_{-\infty}^{m_a} \psi\,dm} \quad . \quad . \quad . \quad (43)$$

Observations of star-counts on photographs taken with different times of exposure will therefore readily decide whether or not the distribution in the cluster is in accordance with the first hypothesis in (42).

A practical test of the criterion (43) was made in the case of the cluster Messier 13, for which suitable counts are given by Pease and Shapley (*Contributions of the Mount Wilson Observatory*, No. 129). The counts refer to a central area of radius 2·09 and consecutive rings of the same width. The areas are denoted by 0, 1, 2, . . . 5, of which the last, however, has been omitted on account of the incompleteness of observations.

Table XV. shows the counts made on seven photographs whose times of exposure are given in the first vertical column.

TABLE XV.

Counts of Stars in Messier 13.

Area. / Time of Exp.	0.	I.	II.	III.	IV.	Adopted Limiting Magnitudes.
m						m
1	518	159	49	23	13	15·7
3	988	359	136	57	43	16·7
6	3276	1618	412	207	119	17·3
15	3620	2602	777	357	175	18·0
22	5026	5116	2177	999	420	18·4
37·5	5035	5923	2852	1263	575	18·8
94	3144	8614	4863	2542	1356	19·6

It must be considered, however, that these counts include the stars of the general system projected on the field of the cluster. If the limiting magnitudes were known, the number of stars not belonging to the cluster could be obtained from Table III. These magnitudes are not stated by the observers, but they may be roughly inferred from the following consideration. The Cape Astrographic plates taken with a 13-inch refractor show with 6^m exposure stars brighter than about 14·0. Theoretically, the 60-inch aperture of the Mount Wilson reflector, for the same time of exposure, should show stars down to $17^m{\cdot}3$. Having fixed this provisional magnitude, those of other exposures may be found from Schwarzschild's formula:

$$m_a - m_b = 2{\cdot}0\,(\log t_a - \log t_b).$$

In this manner the limiting magnitudes in the last vertical column of Table XV. were calculated. They are admittedly uncertain, but probably sufficiently accurate for at least a rough estimate of the number of non-cluster stars included in the counts of Table XV.

From Table III., which represents the average conditions of the general system in medium galactic latitudes (the latitude of the cluster is 40°), we may now calculate the number of stars projected into the areas of the different rings. They are shown in Table XVI.

TABLE XVI.

Number of Stars of the General System in the Field of M. 3.

Limiting Magnitudes.	Area.				
	o.	I.	II.	III.	IV.
15·7	2	5	9	12	15
16·7	3	9	16	22	28
17·3	4	13	22	31	40
18·0	6	19	32	45	58
18·4	8	24	41	56	73
18·8	10	31	53	73	93
19·6	15	43	72	100	129

Subtracting Table XVI. from XV. we finally obtain the numbers of stars actually belonging to the cluster. These are shown in Table XVII.

TABLE XVII.

Number of Cluster *Stars in Messier* 13.

Exp.	Limiting Magnitudes.	Area.				
		o.	I.	II.	III.	IV.
m						
1	15·7	516	154	40	11	−2
3	16·7	985	350	120	35	15
6	17·3	3272	1605	390	176	79
15	18·0	3614	2583	745	312	117
22	18·4	5018	5092	2136	943	347
37·5	18·8	5025	5892	2799	1190	483
94	19·6	3129	8571	4791	2442	1227

These figures show at first sight that the condition (28) is by no means satisfied. Compare, for instance, the ratios $\frac{n_b}{n_a}$ for exposures 1^m and $37^m{\cdot}3$.

Area.	n_b/n_a.
o	10
I.	38
II.	70
III.	108

Obviously, the rate at which the numbers increase with increasing time of exposure is much greater in the outer rings than near the centre, while according to hypothesis it should be the *same for all rings*.

The result shows conclusively that the assumption of a *uniform* luminosity-law and a frequency which depends on the distance d from the centre does *not* satisfy the observed star-counts.

Let us next examine whether more satisfactory results are

obtained with the second hypothesis of (42). We assume, accordingly, the total number of stars in unit volume, f_0, to be the same at all points of the cluster, while the luminosity-law is now

$$\phi(m - \overline{m}),$$

where $\overline{m}$ is a function of the distance from the centre.

An investigation on the basis of this assumption has led to the result that the observed numbers (Table XVII.) in any ring, denoted by the affix x, can indeed be well represented by an equation of the form

$$n_x = f_0 a_x \mathrm{D} \int_{-\infty}^{m} \phi(m - \overline{m}_x)\, dm \quad . \quad . \quad . \quad (44)$$

where a_x is the area of the ring, D the depth of the cluster (found to be the same for all rings) and $\overline{m}_x$ is a function of the radius of the ring.

Let the luminosity-law be such that the numerical values of

$$\log \int_{-\infty}^{m} \phi(m - \overline{m}_x)\, dm$$

for the argument $(m - \overline{m}_x)$ are represented by Table XVIII.

TABLE XVIII.

$(m-\overline{m}_x)$.	$\log \int \phi dm$.	$(m-\overline{m}_x)$.	$\log \int \phi dm$.	$(m-\overline{m}_x)$.	$\log \int \phi dm$.
+0·6	−0·20	−2·0	−0·89	−4·6	−2·32
+0·4	−0·24	−2·2	−0·97	−4·8	−2·45
+0·2	−0·27	−2·4	−1·06	−5·0	−2·58
0·0	−0·30	−2·6	−1·15	−5·2	−2·71
−0·2	−0·33	−2·8	−1·25	−5·4	−2·84
−0·4	−0·37	−3·0	−1·35	−5·6	−2·97
−0·6	−0·41	−3·2	−1·46	−5·8	−3·11
−0·8	−0·46	−3·4	−1·57	−6·0	−3·24
−1·0	−0·52	−3·6	−1·69	−6·2	−3·38
−1·2	−0·58	−3·8	−1·82	−6·4	−3·52
−1·4	−0·65	−4·0	−1·95	−6·6	−3·68
−1·6	−0·73	−4·2	−2·08	−6·8	−3·84
−1·8	−0·80	−4·4	−2·20		

Adopting

$$f_0 \mathrm{D} = 676$$

and the following numerical values of a_x (in sq. minutes of arc) and $\overline{m}_x$:

Ring.		0.	I.	II.	III.	IV.
a_x	=	13·7	41·2	68·6	96·1	123·5
$\overline{m}_x$	=	18·5	20·3	21·6	22·5	23·2

the numbers n_x can be calculated from (44) if the limiting magnitudes m are known.

Table XIX. shows the computed numbers alongside with the observed numbers of Table XVII. The limiting magnitudes which must be adopted to secure the best possible representation of the observed figures are shown in the first vertical column. They agree fairly well with the provisional magnitudes derived from the times of exposures (Table XVII.).

TABLE XIX.

Ring. / Limiting Magnitudes.	O.		I.		II.		III.		IV.	
	O.	C.	O.	C.	O.	C.	O.	C.	O.	C.
15·7	516	525	154	132	40	31	11	9	[−2]	3
16·4	985	1090	350	355	120	94	35	32	15	13
17·5	3272	2880	1605	1550	390	447	176	170	79	76
17·9	3614	3630	2583	2450	745	795	312	309	117	141
18·6	5018	4900	5092	4790	2136	2140	943	851	347	400
18·8	5025	5250	5892	5760	2799	2630	1190	1100	483	525
19·4	(3129)	6310	8571	8910	4791	5130	2442	2570	1227	1230

With the exception of the centre area at longest exposure, th star-counts of which are doubtless incomplete on account of overlapping, the agreement between the observed and calculated numbers may be considered sufficiently close.

The luminosity-integral $\int_{-\infty}^{m} \phi(m-\overline{m}_x)\,dm$ of Table XVIII. is well represented by an error law of the form

$$\int_{-\infty}^{m} \phi(m-\overline{m}_x)\,dm = \frac{h}{\sqrt{\pi}} \int_{-\infty}^{m} e^{-h^2(m-\overline{m}_x)^2}\,dm\,;\quad h = 0{\cdot}3974 \quad . \qquad (45)$$

as is shown by comparison of the figures of Table XX.

TABLE XX.

$(m-\overline{m}_x)$.	$\int_{-\infty}^{m} \phi(m-\overline{m}_x)\,dm$. Table XVIII.	Eqn. (45).
+0·6	0·631	0·632
0·0	0·500	0·500
−0·6	0·389	0·368
−1·2	0·263	0·250
−1·8	0·158	0·156
−2·4	0·0891	0·0887
−3·0	0·0447	0·0459
−3·6	0·0200	0·0215
−4·2	0·00832	0·00912
−4·8	0·00355	0·00349
−5·4	0·00148	0·00120
−6·0	0·00058	0·00037

The investigation of cluster M. 13, the results of which are supported by the evidence from other clusters, therefore confirms the conclusion derived from the general system, that the luminosity-law is approximately represented in the form of an error curve. The individual luminosities are distributed round a mean luminosity $\bar{M}_x = \bar{m}_x - 5 \log r_0$ which depends on the distance d from the centre. Its values for the various rings are

		0.	I.	II.	III.	IV.
$\bar{M}_x$	=	−1·7	+0·1	+1·4	+2·3	+3·0,

assuming Shapley's value of the distance of the cluster of the sun, $r_0 = 11{,}100$ parsecs. The modulus $h = 0{\cdot}397$ is in fair agreement with that derived for the general system: $h = 0{\cdot}315$.

We have shown that the distribution of the stars in a cluster cannot be reconciled with the first assumption (42), *i.e.* with a luminosity-law independent of the positional co-ordinates. It is necessary to introduce a luminosity-law in which the average luminosities appear as functions of the position. This conclusion is in accordance with the result of our investigation regarding the general system, and must be considered a strong argument against the first hypothesis (42).

§ 6. *The Dependence of the Parameters a on Galactic Latitude.*

Table X. shows that of the two quantities a and A, which in the general statistical equations were defined as functions of the spherical co-ordinates λ, β, only the former indicates a pronounced dependence on the galactic latitude β.

Reverting to the two alternative assumptions regarding the density- and luminosity-laws expressed in (42), it is seen at once that the quantities a define the loci of equal star-densities in the first assumption and the loci of equal absolute luminosities in the second assumption.

In the first hypothesis, the density-law within any cone was defined by a function of the form $f(ar)$. It was shown further that the statistical data satisfy the condition $ar = a'r'$, the quantities on the right-hand side referring to a selected standard cone.

Hence taking in the standard cone any special value of $f(a'r')$, the same density in any other cone will be found at distance

$$r = \frac{a'}{a} r'.$$

The density, ν, say, which in the standard cone occurs at distance $r' = 1$ will in any other cone be found at distance $r = a'/a$.

Let fig. 3 represent a section through the general system at right angles to the galactic plane, the sun being at the point O. By means of the numerical values of a'/a shown in Table X. the distances from O at which the density, ν, occurs in different galactic latitudes can be readily calculated. The two curves drawn through the calculated points represent the loci of the density ν obtained

from the material derived from visual and photographic magnitudes. But if the hypothesis were correct, the two curves should obviously be identical, since ν, which represents the *total* number of stars in unit volume, cannot be supposed to depend on the *optical* conditions on which the statistical data are based. Hence, if these latter are reliable, the first hypothesis must doubtless be rejected.

In the second hypothesis, the same curves represent the loci of equal average absolute magnitudes, $\bar{M}_v$ or $\bar{M}_p$, according as to whether the apparent magnitudes are defined on the visual or photographic scales. In this hypothesis the luminosities are supposed to be functions of the positional co-ordinates while the density is a constant throughout the system. Changes of bright-

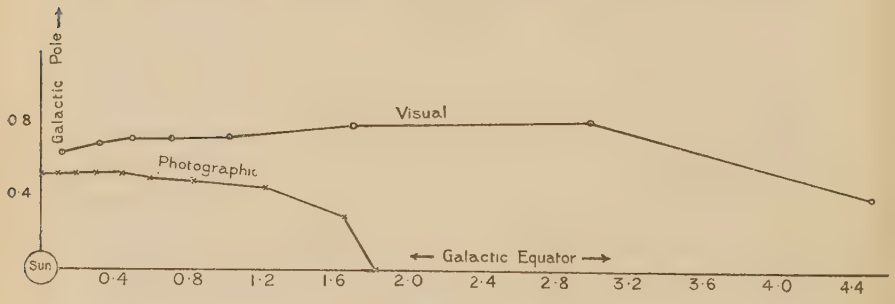


Fig. 3.

ness, however, are associated with changes of the colour-index. Hence the difference $(\bar{M}_p - \bar{M}_v)$ is a function of the positional co-ordinates, *i.e.* the loci of $\bar{M}_p$ and $\bar{M}_v$ are represented by *different* curves, depending on the effective wave-lengths for which the apparent magnitudes have been determined. This conclusion is therefore in agreement with the evidence of fig. 3.

The absolute brightness of a star depends in general on two conditions, viz. the surface brightness and the size. In addition, considering that the light has travelled across vast distances, the possibility of a loss of radiant energy through scattering by intervening cosmic dust must be taken into consideration.

Generally, therefore, the functional relation between $\bar{M}$ and the positional co-ordinates ρ, λ, β must be supposed to be due to the combined effect of these three agencies.

Fig. 3 shows that for latitudes between the poles and about $\pm 20^\circ$ the curves of equal luminosity are approximately straight lines parallel to the galactic equator. Within this region the surfaces of equal luminosities are therefore approximately planes parallel to the galactic plane. In other words, the statistical data can be explained on the assumption that the luminosities of the stars are arranged in plane-parallel layers, the average absolute magnitude at any point P being analytically expressed in the form

$$\bar{M} = -A + a.d$$

where A and a are quantities depending on the effective wave-lengths only, and d is the distance of P from the galactic plane.

This simple structure, however, breaks down in low latitudes. Obviously, if the luminosities are arranged in layers parallel to the galactic equator, a in the galaxy should be zero, *i.e.* the average magnitudes $\bar{M}$ should be constant at all distances. Table X. shows clearly that this condition is not fulfilled. Hence, either the surfaces of equal intrinsic luminosities cease to be planes, or some cause not inherent in the stars themselves makes itself felt, through which distant stars are made to appear relatively fainter than near stars. This raises the question of the existence of space absorption.

If the light of a star during its travel through space is passing through a uniformly distributed scattering medium, the magnitude is increased in proportion to the distance, *i.e.*

$$M = M_i + \gamma\rho$$

where M_i is the intrinsic absolute magnitude and γ represents a coefficient depending on the density of the scattering medium and on the effective wave-length. The ratio γ_p/γ_v for photographic and visual light, is 2·5 approximately, according to observations made in regard to the extinction caused by our atmosphere, and is in satisfactory agreement with the theoretical value derived by Lord Rayleigh. Now we find from Table X. for the galaxy $\frac{a_p}{a_v} = \frac{0^m55}{0^m22} = 2{\cdot}5$, *i.e.* exactly the value we should expect if a were entirely due to space absorption. This agreement is in favour of the view that in the galaxy, so far as the intrinsic brightnesses are concerned, $a = 0$, *i.e.* that the intrinsic luminosities are the same at all distances, a result we should obtain if the luminosities are arranged in plane layers parallel to the galactic equator.

The result is in agreement with the deductions drawn by Shapley from his study of globular clusters, though at first sight Shapley's conclusions appear to contradict the assumption of space absorption. He found that the comparison between distant and near clusters showed no evidence of selective absorption which could be ascribed to this cause. He therefore rightly infers that in the regions of the sky occupied by clusters space absorption is practically non-existent. But when the distribution of clusters is examined, the remarkable fact becomes apparent that no such formations are found in the close proximity of the galaxy. Proceeding from the galactic poles to lower latitudes the frequency rises rapidly, reaches a maximum at about $\beta = \pm 7^\circ$ whence it almost suddenly drops to zero. No globular cluster has so far been discovered within the galactic belt $\pm 5^\circ$. The assumption of absorbent cosmic matter confined to a comparatively narrow equatorial galactic belt would readily explain this phenomenon, just as it explains the deformation of the planes of equal luminosities in the general system.

With increasing improvement in the accuracy of the statistical data it may eventually become possible to separate the portion of a due to intrinsic light properties of the stars from that caused by the influence of scattering. At present such an attempt is scarcely warranted.

What is claimed here to have been proved is the fact that a, if not entirely, is at least mainly due to *optical* agencies. The assumption of a *uniform* luminosity-law cannot therefore be reconciled with the observational evidence, and hence the distribution of star matter derived from this assumption is open to grave doubt. Unless the observed facts are far less reliable than we have reason to believe, our conceptions of the structure of the universe have to be radically changed. Wherever observational evidence has placed us in a position to distinguish between the extreme conceptions, viz. homogeneous distribution of luminosities and heterogeneous distribution of star matter on the one hand, and heterogeneous distribution of luminosities combined with a homogeneous distribution of star matter on the other, the verdict has always been in favour of the latter assumption. It seems thus hardly possible to evade the conclusion that the apparent heterogeneity of the universe is largely an optical phenomenon. Stars in the galaxy appear to our eye more frequent than in high latitudes, not because they are more densely crowded but because they possess greater luminosities. Judging by the evidence here discussed, the commonly held conception of the stars round the sun forming an island universe separated from other similar systems by vistas of space devoid of star matter is at least open to serious doubt, and certainly will have to be rejected if more numerous and refined observations confirm the statistical data on which the present investigation is based.

On the Distribution of the Stars down to the 11th Magnitude. (A reply to Dr. Nort.) By Dr. A. Pannekoek.

(*Communicated by the Secretaries.*)

In our communication on the distribution of the stars to the 11th magnitude (*Monthly Notices*, 79, No. 5) we have drawn attention to the fact that the method used by Mr. Henie and Dr. Nort for the derivation of the reduction to centre must give too large values for the plates which have their centres in the Milky Way. Dr. Nort has since re-examined this reduction for the galactic plates, by correcting all densities for difference of galactic latitude. Indeed he now finds for these plates a reduction to centre clearly smaller than in his first research, though still greater than for the plates at higher latitude, that gave the same reduction to centre as formerly.

According to his communication in the *Monthly Notices*, 79, No. 9, Dr. Nort believes his first erroneous reduction to centre has also been the cause of the systematic differences I found between adjacent Harvard plates. If he was right, these differences should only occur, or at least should be greatest, at the limits of the dense galactic and the poor extra-galactic plates. A glance at the coefficients I deduced (*cf.* p. 334) shows this is not true; thus we find great differences between the plates 15 and 16 (ratio of densities 2·0), between 15 and 27 (ratio 2·7), 29 and 30 (ratio 2·0), 18 and 30 (ratio 2·7), 14 and 26 (ratio 2·4) that cannot be attributed to erroneous reductions to centre, as they are all plates with small or moderate density lying outside the Milky Way. Moreover, the new reduction to centre does not much deviate from the old: as for border stars lying 90–130 mm. from the centre, this deviation is only 0·10 to 0·16 magn. (old reductions $0^m{\cdot}62$ to $0^m{\cdot}95$, new reductions $0^m{\cdot}52$ to $0^m{\cdot}79$). The error caused by it in the reduction of density to centre, taken as a logarithm, is not more than 0·07, corresponding to a ratio of 1·18. By this quantity our results for the dense galactic plates should be corrected, and such differences could be explained in the way suggested by Dr. Nort. But as it is much smaller than the plate coefficients I found, these cannot be explained in this way, and they will not be materially changed by the new results of Dr. Nort.

Dr. Nort finds another ground of criticism in the fact that in my reduction of adjacent plates to each other I could only make use of the border portions, that are the worst parts of the plates. Generally speaking, he is certainly right; but the question is, how great the errors thus introduced may be? In this we need not restrict ourselves to general statements, as by the extent of their mutual agreement the different results themselves show the degree of their uncertainty. Now, we have stated already: "As the mean error of a result of log C is 0·05, and as 35 out of 55 values are 0·10 or more, and 10 values are 0·20 or more, the reality of the majority of these corrections stands beyond doubt" (p. 334). This means, that though I was obliged to use the worst parts, the differences between the limiting magnitude and mean densities of different plates are so great, that their neglect leaves much greater errors than are introduced by the comparison of border parts.

To test this I have still used another method. In my research I have computed the mean density for each of the four quadrants of each plate and its logarithmic deviation from a normal spheroidal star system (*cf.* Table II., last column). The mutual differences of these four values belonging to the same plate do not depend on the plate coefficients used, and the mean error of a logarithmic deviation computed from them gives a measure for the real irregularities at the sky. When we take, however, four quadrants lying on four plates around their common edge, in their mutual differences the systematic plate-errors play a rôle. Using in this way all the edges on declination $+15°$ and $-15°$, and all the plates at declination $+30°$, $0°$, $-30°$, I found for the mean error

from quadrants on the same plate $\sqrt{0{\cdot}0077} = 0{\cdot}09$

from adjacent plates, numbers of Nort, uncorrected $\sqrt{0{\cdot}0149} = 0{\cdot}12$

from adjacent plates, corrected $\sqrt{0{\cdot}0086} = 0{\cdot}09$

This proves in still another way that the coefficients I applied really ameliorate the data of Dr. Nort, and that by applying them the greater part of the irregularities due to the plates are removed.

Dr. Nort seems to believe that his criticism also throws doubt on my results as to the centres of paucity or funnel-shaped holes in Taurus and Ophiuchus; so I must emphasise that they are confirmed by most catalogues and charts that are able to show the general distribution of the stars, such as the Bonner Durchmusterung and the catalogue plates of the Carte du Ciel as well as the chart plates so far as they have been published.

Monochromatic Photographs and Photometric Measures of the Dumb-Bell Nebula (N.G.C. 6853). By J. H. Reynolds.

This nebula, although it is not usually included amongst the planetaries, has undoubtedly the principal characteristics of the class. These characteristics may be summarised as—

1. A gaseous emission spectrum.
2. A definite circular, elliptical, or compound elliptical outline.
3. The presence of a central star.

All these are represented in the nebula, and it should be therefore regarded as a true planetary, notwithstanding its large size.

The spectrum is unusually simple for a nebula of this class, as it appears to be entirely concentrated visually in the nebulium radiation at λ 5007 and photographically in the radiation λ 3727, the hydrogen radiations being either altogether absent or so faint as to be imperceptible visually or photographically. The nebulium radiations also at λ 4958 and λ 4363 are probably not represented.

Visual observations of the spectrum were made on several occasions with the 28-inch reflector and a Hilger direct-vision slit spectroscope of considerable dispersion. Only the strong line at λ 5007 was seen, no other lines being even suspected. An exposure of two hours with a two-prism spectrograph on a panchromatic plate yielded no lines at all. In this case the radiation at λ 3727 was not included on the plate, and the insensitiveness of the plate to the blue green was no doubt responsible for the absence of λ 5007. Von Gothard, however, records an exposure of 90 minutes with an objective prism which yielded a strong monochromatic image at λ 3727 alone, as if no prism had been interposed in front of the lens.*

* *Ast. Nach.*, No. 3738, "Die Entstehung der photographischen Aureole um die Nova (3, 1901) Persei."

The absence of the usual nebulium series is remarkable, and raises the question as to the physical conditions necessary for its full development.

Another point of great interest is the relationship of this spectrum to that of new stars in their later stages. Father Cortie drew attention, in his communication to the B.A. in September, to the fact that in 1919 July and August the spectrum of Nova Aquilæ III. was concentrated visually in the radiation λ 5007 only,* and had the aspect of a small planetary nebula, and there are other instances where this radiation is said to be alone in producing the visual image of planetary nebulæ.

There is, besides, the grouping of planetary and annular nebulæ in and near the galactic plane, which is also the same for novæ.

The division between ring and planetary nebulæ seems to be entirely an arbitrary one, and an examination of the distribution of the 57 planetary and ring nebulæ in the N.G.C. shows that 49 are either in or near the Milky Way.

We have therefore a good deal of evidence which supports the hypothesis that planetary nebulæ are really the later and more permanent forms of novæ outbursts, an explanation of their origin which was suggested by the ring of nebulous matter which developed around Nova Persei. It is not necessary to suppose that the nebula was pre-existent round the star, as the enormous velocities which are indicated by line displacements would rather support the view that the hydrogen and nebulium were expelled from the star itself, the spheroidal shells of gas being the result of separate pulses of ejection.

It has been suggested by Arrhenius that the gaseous nebulæ shine by the radiation of negative electrons from the stars. If this is the case, the central star of a planetary nebula is the source of radiation, and the nebula would be visible as long as the star continued to radiate.

It is probable on this hypothesis that some planetary nebulæ are still expanding; at any rate, there seems to be no reason why an expanding nebula should not be regarded as a possibility.

Monochromatic Photographs.

The methods adopted for isolating the monochromatic images at λ 5007 and λ 3727 were the same as those previously adopted for the Orion Nebula (*M.N.*, **79**, 8).

Owing to the absence of the hydrogen radiations there was no need to cut out the blue region for the nebulium image, and the æsculine screen, which transmits the whole of the visual spectrum without substantial reduction, was employed to eliminate λ 3727. As the blue-green nebulium radiation λ 5007 is comparatively ineffective with an ordinary fast photographic plate, no screen was used for the radiation λ 3727. It is worth noting that the photographs hitherto secured of this nebula have been due almost

* λ 4363 was, however, present in the photographic spectrum.

entirely to this ultra-violet radiation, and the visible image has therefore not been previously photographed.

Experiments were carried out with various plates and colour screens to determine the best method of securing an image in the nebulium radiations at λ 5007, as the blue green is the most difficult to deal with in photography of the visual spectrum owing to the comparative insensitiveness of all kinds of plates to this region. Rapid ordinary plates yield an image after prolonged exposure, but the rapid isochromatic plates, although especially sensitive in the yellow green, were found to be much more effective than either these or panchromatic, the most suitable being the Wellington Anti-screen.

At the same time, in order to get the best results in this work, it may be found better to bathe plates specially for this region, as the exposures necessary for an image of sufficient density are much longer than would be expected from the visual brightness.

The following is a list of the exposures made:—

Date.	No.	Plate.	Screen.	Exposure.
1919				
Sept. 26	1	Flashlight	Unscreened	50 mins.
,, ,,	2	,,	Æsculine	60 mins.
,, 27	3	Super Speed Ortho.	,,	1 hr. 45 mins.
Oct. 18	4	Flashlight	Unscreened	35 mins.
,, 21	5	,,	,,	1 hr. 45 mins.

Plate No. 2, which had an effective range from λ 4000 to the usual limit in the blue green for ordinary fast plates, gave a faint image of the nebula. The strength of λ 5007 is quite sufficient to account for this result. The exposures of the screened and unscreened plates corresponding to λ 5007 and the radiation λ 3727 after careful comparison showed no difference either in extension or regional brightness. This is not in agreement with the results obtained by the slitless spectrogram method on the small bright planetary and annular nebulæ, where nebulium gives the smaller image, as it does also in the Orion Nebula. All the exposures agree in showing a dark space round the central star, which is not a photographic effect, as other stars on the nebula are normal. This central star is about equal in magnitude to the small star south-east of the nebula. The latter is easily seen visually, but no glimpse is to be caught of the central star with an eyepiece, although it is conspicuous on Plate No. 3, in which the ultra-violet light is cut out: it seems as if the presence of nebulosity renders the eye incapable of detecting small stars involved in it.

Micrometrical and Photometrical Measures.

The outline of the nebula is a combination of two ellipses, the axes crossing almost at right angles. The minor axis of the smaller and brighter passes through the central star in a direction from 65°

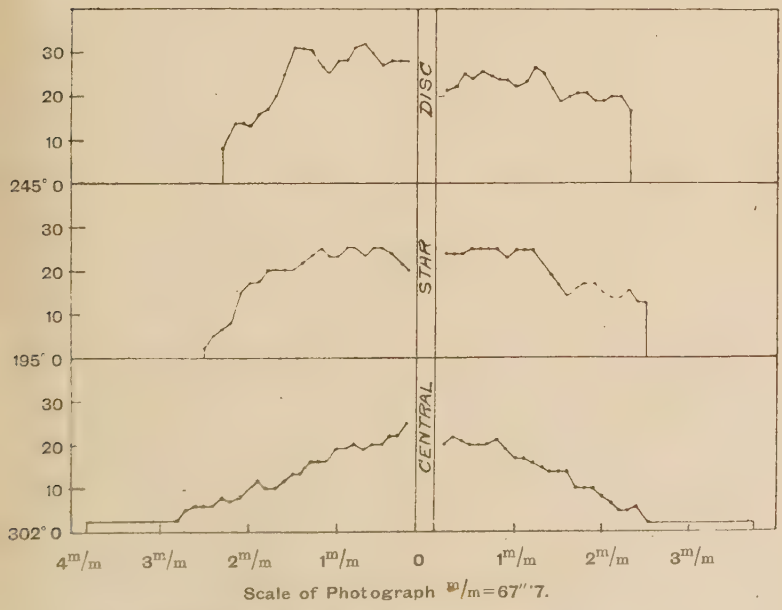
to 245°. This ellipse has a bright edge which extends for some distance each side of the axis, and is consistent with the view that in its main structure the nebula consists of a hollow gaseous spheroid or ellipsoid. There are considerable irregularities of brightness in the nebula itself which are much more prominent in photographs than visually.

Micrometrical and photometrical measures of the image on Plate No. 4 (35 mins exposure) were made with the photomicrometer. The negative was placed and centred on a divided metal disc which could be rotated, thus bringing any required diameter of the image in the direction of the micrometer screw. The central star disc was taken as the fiducial point, and measures of these diameters gave the following results:—

Star − 245° 2′ 35″·7 „ − 65° 2′ 35″·7	Total diameter 5′ 11″·4.
Star − 195° 2′ 49″·15 „ − 15° 2′ 49″·15	Total diameter 5′ 38″·3.
Star − 122° 4′ 20″ „ − 302° 4′ 16″	Total diameter 8′ 36″.

At the same time measures of the photographic densities were made at intervals of ·1 mm. along these diameters.

Light Distribution in the Dumb-Bell Nebula (N.G.C. 6853).



Scale of Photograph $^{m}/_{m}$=67″·7.

The densities were calibrated with actual light values, by cutting a plate of the same kind as the negative into ten pieces,

and exposing them simultaneously to a faint light at distances calculated from the inverse square law.

The densities of the pieces were then calibrated to the densities of the arbitrary scale of the photomicrometer and the resulting values plotted on a curve, the abscissæ of which were the divisions of the arbitrary scale, and the ordinates, an adopted scale of light units, gave 100 units at a distance of 1 foot from the light.

The light value of the density constant of the plate was deducted from the values given by the density curve before plotting the light values of the three diameters from the measures. The results are only given in the diagram, as the numerical work involved is not of sufficient importance for publication.

Notes on a Disturbed Sun-spot Area on the Sun's Eastern Limb, 1919 May 29. By Rev. A. L. Cortie, S.J.

The question has been raised, and deserves careful consideration, whether the Einstein deflection of the light of the stars passing the sun, 1·75″ at the limb of the sun, may not be due to electrons which have been projected into the regions surrounding the sun from foci of activity situated on the sun. What would be the effect of such clouds of electrons upon a beam of light passing through them, which is itself an electro-magnetic phenomenon? The sun, too, is itself a great magnet.

In former papers * it has been shown that streamers in the solar corona have been closely connected with sun-spot and faculous areas of long-continued activity. To quote from my paper, "The Mode of Propagation of the Sun's Influence in Magnetic Storms," † "it is reasonable to suppose, from the investigations of Dr. Nicholson and others, that such rays are the carriers of negatively charged electrons. As these particles recede into space from the sun they would mutually repel one another, and the result would be the formation of clouds of particles, as postulated by Father Sidgreaves in his paper "On the Connection between Solar-spots and Earth-magnetic Storms." ‡ The double curvature in some of the streamers of solar coronæ, giving a trumpet-shaped projection, is evidence of such mutual repulsion. Such clouds would more abound in the sun-spot regions, and might reach large dimensions by the cumulative action of successive accretions."

In the case of the total solar eclipses of 1893, 1898, 1905, 1908, and 1914, sufficient evidence was adduced to prove that long-enduring belts of sun-spot activity were associated in the coronæ with diverging sheaves of coronal streamers. In the four

* *Monthly Notices*, 73, 431–436, 539–543; 75, 496–501.
† *Ibid.*, 73, 542.
‡ *Memoirs*, 54, 91, 92.

eclipses first named the regions of solar activity happened to be situated near to the sun's limb on the dates of total eclipse.

In the recent eclipse such a disturbed region extending from about longitude 356° to 30°, and from 0° to 25° N. latitude, was just on the sun's limb on the day of the eclipse. Pending a fuller detailed description of the life-history of this disturbance, the following points may be noted. The disturbance lasted from April 6 to August 24, and was active at the time of solar eclipse. It commenced by a few isolated bright clouds of faculæ on April 6. Sun-spots surrounded by bright faculæ were seen on May 3, at the next solar rotation, which developed greatly between May 7 and May 13. At the following rotation, on May 30, the faculæ were flowing northward from the main spot, and a new spot had formed in longitude 8°; N. latitude 18°. On June 1 yet other spots had formed in bright faculæ in longitude 2°; N. latitude 25°, which had developed into a group by June 7. On June 29, at the next rotation, the faculæ were still flowing northward, and apparently backward, in accordance with Carrington's law. At the succeeding rotation the stream of faculæ had reached N. latitude 40°. On August 24, after another solar rotation, a few scattered clouds remained.

In the sun's corona over the position of the disturbance there was a sheaf of bright rays, one of the brightest being in the more northerly position. The great prominence was also bending over from the strong upcast at latitude 40° S. to the region of the sun-spot disturbance. Including the prominence and the faculous region, the sun's E. limb, on May 29, was disturbed from 40° S. to 25° N.

Stonyhurst College Observatory:
1919 *December* 10.

The Spectrum of Nova Aquilæ, 1919 *July, August.*

By Rev. A. L. Cortie, S.J.

The star was observed telescopically with the 15-inch equatorial, with powers of 250 and 500, on 1919 July 25, 29, August 5, 6, 14. Its appearance was that of a bright stellar nucleus surrounded by a circular disc of light. Its colour was of a greenish tint. According to observations kindly furnished by Mr. C. F. Butterworth, the magnitude of the star on July 25 was 6·84, on August 7, 7·24, and on August 10, 7·04.

When its spectrum was examined visually with a McClean star spectroscope, almost the whole of the light was concentrated in one bright green ray, evidently λ5007. Two fainter radiations were glimpsed, one in the blue, probably λ4363, near H_γ, and one in the yellow, perhaps D_3. H_α could not be seen. These radiations were superposed upon a faint continuous spectrum.

On August 6 and 14 photographs of the spectrum were obtained with the Whitelow prismatic camera, consisting of a prism of 19° angle and 6 inches edge, in combination with a Dallmeyer portrait lens of 6 inches aperture and 30 inches focal length. The photographs were made on Ilford Rapid Isochromatic plates, the exposure on August 6 being from 9.35 to 10.5 p.m. G.M.T., and on August 14 from 10.0 to 11.0 G.M.T. The trails were so slow that the images were almost stationary. The spectral radiations on the photographs are round, the images of a luminous disc.

The spectrum from λ3820 to λ5150 covers 8 mm. linearly on the plates. An error of only ·01 mm. in the micrometer setting would give rise to a mean error of nearly 2 Angström units in the wave-lengths. Several of the lines were also difficult to measure on account of their faintness. The radiations being broad, the micrometer wire may not have been placed exactly at their centres. Nevertheless it is not difficult from the measures to assign the probable origins to the majority of the radiations. The following are the wave-lengths deduced from the measures:—

Wave-Lengths.		Intensity.	Probable Origin.
Aug. 6.	Aug. 14.		
...	3839	1	3836 H_η.
3868	3861	2	3868 Nebular.
3883	3891	2	3889 $H\zeta$.
3925	...	1	?
...	3949	2	?
...	3963	2	3965 Nebular.
3970	...	2	3970 H_ϵ.
4102	4098	3	4102 $H\delta$.
4363	4363	10	4363 Nebular.
...	4561	1	4555 Wolf-Rayet stars.
...	4942	3	4940 ,, ,,
5007	5007	10	5007 Nebular.

On the photographs the radiations λ5007 and λ4363 are very much the brightest. On account of the small dispersion the nebula line λ4959 probably coalesces with λ5007. H_β was not recognised as being present. There was a faint continuous spectrum.

Stonyhurst College Observatory:
1919 *December* 5.

On Professor Howe's Method of solving Kepler's Equation.

By H. C. Plummer.

1. The method which Professor A. Howe proposed * in 1880 for solving Kepler's equation, and to which he recalled attention at the recent meeting † of the American Astronomical Society, deserves to be better known than it seems to be. It rests on an approximation which is very simply obtained. Kepler's equation being written in the form

$$E - M = e \sin E \quad . \quad . \quad . \quad . \quad (1)$$

it is evident that the equation

$$\sin (E' - M) = e \sin E' \quad . \quad . \quad . \quad (2)$$

will give an approximation E′ which differs from E by the third order, when the eccentricity e is regarded as a small quantity of the first order. This can be written in the well-known form—

$$\tan (E' - \tfrac{1}{2}M) = \frac{1+e}{1-e} \tan \tfrac{1}{2}M \quad . \quad . \quad . \quad (3)$$

which is the most convenient for calculating E′. Let

$$2\eta = (E' - M) - \sin (E' - M) \quad . \quad . \quad . \quad (4)$$

Then (1) and (2) give immediately

$$E' - E - 2\eta = e(\sin E' - \sin E)$$

or

$$E' - E = \frac{2\eta}{1 - e \cos \{E' - \tfrac{1}{2}(E' - E)\} \sin \tfrac{1}{2}(E' - E)/\tfrac{1}{2}(E' - E)} \quad (5)$$

This equation is accurate. But clearly to the third order $E' - E = 2\eta$. Hence the result of substituting this value on the right-hand side is to give the approximation

$$E' - E = \frac{2\eta}{1 - e \cos (E' - \eta)} \quad . \quad . \quad . \quad (6)$$

which is evidently correct to the seventh order. This is Professor Howe's approximation, and he has shown that the error contains in fact $e^2\eta^2$ as a factor, which is of the eighth order. The approximation obtained by the use of (3), (4) and (6) is therefore very high.

2. Let

$$E = E' - F.$$

* *A.N.*, 2322. † *Popular Astronomy*, 27, 569.

Then (5) becomes

$$F = \frac{2\eta}{1 - e\cos E' \sin F/F - e\sin E'(1-\cos F)/F}$$

$$= \frac{2\eta}{1 - e\cos E'(1 - \frac{1}{6}F^2 + \dots) - e\sin E'(\frac{1}{2}F - \frac{1}{24}F^3 + \dots)}$$

$$= \frac{2\eta}{1 - e\cos E'}\left[1 + \frac{\sin E'}{2(1-e\cos E')}\cdot eF + \frac{3e - 2\cos E' - e\cos^2 E'}{12(1-e\cos E')^2}\cdot eF^2 + \dots\right] \quad . \quad . \quad . \quad . \quad (7)$$

and the terms not written down are at least of the thirteenth order. It is now easy to make approximations. Beyond E', correct to the second order, and E' − 2η, correct to the third, the next approximation is given by

$$F = \frac{2\eta}{1 - e\cos E'} \quad . \quad . \quad . \quad . \quad (8)$$

which is already correct to the sixth order, whereas Howe's approximation (6) is correct to the seventh order. Hence, at the next step,

$$F = \frac{2\eta}{1 - e\cos E'}\left[1 + \frac{\sin E'}{(1-e\cos E')^2}\cdot e\eta\right] \quad . \quad . \quad (9)$$

is correct to the ninth order, and the error of the tenth order, which depends entirely on the term in F^2, is

$$\epsilon = -\tfrac{4}{3}\cos E' \,.\, e\eta^3.$$

Since, approximately, $\eta = \frac{1}{12}(E' - M)^3 = \frac{1}{12}e^3 \sin^3 E'$, and the greatest value of $\cos E' \sin^9 E'$ is about 1/5, ϵ cannot exceed $e^{10}/6500$ numerically, or $32''\,.\,e^{10}$. Thus even when $e = \frac{1}{2}$, and special methods are available for really large values of e, the error of (9) is scarcely likely to reach $0''{\cdot}04$. But when e is as large as $\frac{1}{2}$ the magnitude of the error cannot be safely judged by its order in e, for terms of a higher order can exceed those of a lower in amount.

3. As the series (7) is highly convergent it may seem tempting to continue the approximations by reversing the series for 2η in terms of F. But for the smaller values of e it is unnecessary to go beyond (8) and (9), the second term of which requires at most a 5-figure calculation, and for the larger values of e the development soon becomes complicated and unsatisfactory. It appears simpler to return to (5), which may be written

$$F = \frac{2\eta}{1 - e\cos(E' - \frac{1}{2}F)\sin\frac{1}{2}F/\frac{1}{2}F},$$

or, in general, since $\frac{1}{2}F$ is not a large angle,

$$F = \frac{2\eta}{1 - e\cos(E' - \frac{1}{2}F)\cos\frac{1}{4}F} \quad . \quad . \quad (10)$$

and to solve this equation by continued approximation. The first approximation F_1 is given by (8), and the result of substituting F_1 in the right-hand side of (10) is to give F_2 correct to the tenth order. The result of substituting F_2 will be correct to the fourteenth order, and so on. Even when order is not in itself a safe guide it will be found that this manner of solving (10), which is only a modified form of Kepler's equation, will lead directly and rapidly to the required result in all cases of elliptic motion. This will be seen best by actual examples.

4. *Example* 1.*—

$$e = 0{\cdot}5549454, \qquad M = 34^\circ\ 19'\ 36''{\cdot}14.$$

$1+e$	0·1917152	e	9·7442503	e	9·7442503
$1-e$	9·6484133	$\cos E'$	9·6364955	$\cos(E'-\eta)$	9 6470194
	0·5433019		9·3807458		9·3912697
$\tan \frac{1}{2}M$	9·4897489	$(e \cos E')$	0·2402956		$=0{\cdot}2461896$
	0·0330508				
$E'-\frac{1}{2}M$	$47^\circ\ 10'\ 41''{\cdot}05$	$1-e\cos E'$	9·8806446	$1-e\cos(E'-\eta)$	9·8772621
E'	64 20 29·12	2η	3·6879488	2η	3·6879488
$E'-M$	30 0 52·98		3·8073042		3·8106867
2η	1 21 14·71	F_1	6416″·59	F'	6466″·76
	$=4874''{\cdot}71$		$=1^\circ\ 46'\ 56''{\cdot}59$		$=1^\circ\ 47'\ 46''{\cdot}76$

The value of 2η above can be taken from Bauschinger's *Tafeln*, No. XL. Howe's approximation is represented by $E'-F'=62^\circ\ 32'\ 42''{\cdot}36$, which is over 16″ in excess. The approximation $E'-F_1=62^\circ\ 33'\ 32''{\cdot}53$ is 66″·74 in excess. The second term in (9) is 65″·72, leaving an error of 1″. We proceed thus:

	$n=1.$	$n=2.$
e	9·7442503	9·7442503
$\cos(E'-\frac{1}{2}F_n)$	9·6502833	9·6504224
$\cos \frac{4}{3}\frac{1}{4}F_n$	−175	−179
x	9·3945161	9·3946548
	$=$ 0·2480368	0·2481160
$1-x$	9·8761966	9·8761508
2η	3·6879488	3·6879488
	3·8117522	3·8117980
F_{n+1}	6482″·64	6483″·33
	$=1^\circ\ 48'\ 2''{\cdot}64$	$=1^\circ\ 48'\ 3''{\cdot}33$

The last number is F_3 and $E'-F_3=62^\circ\ 32'\ 25''{\cdot}79$, which is the true value of E within the capacity of 7-figure tables. It is evident

* Bauschinger's *Tafeln*, p. 6.

that 6-figure tables would have sufficed after calculating E'. In ordinary practice one figure less would be required in such an example, so that four logarithms would be found in 6-figure and the rest in 5-figure tables.

5. It is needless to give an application to a small eccentricity, all cases as far as $e = 0{\cdot}25$ being covered by Howe's method without modification. This satisfies the needs of the majority of the minor planets. But an extreme illustration will now be given.

Example 2.—

$$e = 0{\cdot}961733, \qquad M = 31^\circ\ 48'\ 55''{\cdot}8.$$

$1+e$	0·2926399	$\sin(E'-M)$	9·9734713	e	9·9830545
$1-e$	8·5828244		5·3144251	$\cos E'$	9·3176045$_n$
	1·7098155		5·2878964		9·3006590$_n$
$\tan \frac{1}{2}M$	9·4548506		194042″·3	$(e\cos E')-$	0·199829
	1·1646661		$= 53^\circ\ 54'\ 2''{\cdot}3$	$1-e\cos E'$	0·0791194
				2η	4·7678546
$E'-\frac{1}{2}M$	$86^\circ\ 5'\ 4''{\cdot}4$				4·6887352
E'	101 59 32·3	2η	16 16 34·2	F_1	48835″·5
$E'-M$	70 10 36·5		$= 58594''{\cdot}2$		$= 13^\circ\ 33'\ 55''{\cdot}5$

Three successive approximations are now made:

	$n=1$.	$n=2$.	$n=3$.
e	9·9830545	9·9830545	9·9830545
$\cos(E'-\frac{1}{2}F_n)$	8·9580817$_n$	8·8952582$_n$	8·8873044$_n$
$\cos\frac{3}{4}F_n$	− 10149	− 12365	− 12637
x	8·9401213$_n$	8·8770762$_n$	8·8690952$_n$
$=$	−0·087121	−0·0753488	−0·0739767
$1-x$	0·0362779	0·0315494	0·0309949
2η	4·7678546	4·7678546	4·7678546
	4·7315767	4·7363052	4·7368597
F_{n+1}	53898″·5	54488″·5	54558″·2
	$= 14^\circ\ 58'\ 18''{\cdot}5$	$15^\circ\ 8'\ 8''{\cdot}5$	$15^\circ\ 9'\ 18''{\cdot}2$

The convergence of the process being now assured, the final adjustment is made by the rule

$$F - F_4 = \frac{(F_4-F_3)^2}{(F_3-F_2)-(F_4-F_3)} = \frac{(69''{\cdot}7)^2}{520''{\cdot}3} = +9''{\cdot}3.$$

Hence $F = 15^\circ\ 9'\ 27''{\cdot}5$ and $E = E' - F = 86^\circ\ 50'\ 4''{\cdot}8$, which is as accurate as the data.

6. This example is chosen to show not that the method is necessarily the most suitable in this particular case, but that it is perfectly general in its application. It is simple and direct, re-

quiring no special tables, and it has no exceptions, whatever the eccentricity (<1) may be. The example is taken from a paper * by the late Mr. W. S. Aldis and was intended to represent the circumstances of Halley's Comet ten years before perihelion. It is employed by him to illustrate the use of an extensive table of the function $C(\theta)=\frac{1}{2}(\theta+\cos\theta)$. But an evil fate often seems to attend illustrative numerical examples, and a worse fate than usual is in evidence here. The (negative) mean anomaly is 170335″·8, correctly stated, and this is 47° 18′ 55″·8. But 31° 48′ 55″·8 is set down, evidently the result of dividing by 60 and 90 instead of 60 and 60. After two pages of approximation the eccentric anomaly 101° 20′ is obtained, which is still 33″ short of the desired result. But this result, 101° 20′ 33″·2, is the eccentric anomaly corresponding to the mean anomaly 47° 18′ 55″·8 and not the erroneous 31° 48′ 55″·8 on which the calculation is based. It is difficult to explain this unfortunate result, but the error clearly lies too deep for simple correction and the whole example should be ignored. The above method depends on the accurate calculation of E and 2η; the rest of the process can be performed with one figure less throughout. Professor Howe desires a table of $\theta-\sin\theta$ to give 2η with the argument $\theta=E'-M$. Such a table would conduce to accuracy as well as rapidity. No. XL. in Bauschinger's collection is convenient but stops at 39°. This suffices for the application of the method to the minor planets. Beyond this point the tables of Mr. Aldis can be used, though they are not very convenient. They give

$$C(\theta)=\tfrac{1}{2}(\theta+\cos\theta)$$

expressed in seconds of arc, over the range $-90^\circ<\theta<90^\circ$. Hence the value of $\theta-\sin\theta$ can be inferred according to the following rules:—

$$\begin{aligned}
0^\circ<\theta<180^\circ,\quad & \theta-\sin\theta=90^\circ-2C(90^\circ-\theta)\\
180<\theta<360,\quad & =270^\circ+2C(\theta-270^\circ)\\
0<-\theta<180,\quad & =-90^\circ+2C(\theta+90^\circ)\\
180<-\theta<360,\quad & =-270^\circ-2C(-\theta-270^\circ).
\end{aligned}$$

A simple table of $\theta-\sin\theta$ would be more generally useful. But the direct calculation of 2η presents no real difficulty, since it is absolute and not relative accuracy which is here required. Perhaps the best compromise between convenience and economy of space would be to extend the existing table of $\theta-\sin\theta$ between 30° and 60°, narrowing the interval of the argument to 20″ so that the differences would not exceed 10″. The range from 60° to 90° should then be supplied by a table of $\sin\theta$ expressed in arc (degrees, minutes and seconds), in which the interval of the argument would begin with 20″, but might be widened afterwards, since between 81′ and 90° an interval of 1′ would offer the same facility of interpolation. Such tables would probably find many uses.

* *M.N.*, 62, 636.

The Short-Period Variable SW Andromedæ. By C. Martin and H. C. Plummer.

1. The variability of SW Andromedæ was discovered by Miss Cannon * in 1906 from an examination of chart plates at Harvard. At the time of the discovery 461 plates, extending over seventeen years, were examined and showed a variation between the approximate limits 8·7 and 9·9, with a period of just under half a day. The discovery number was 5·1907 and the number on the Harvard system is 001828. Other designations are BD + 28°·54 and A.G.C. (Cambr. E.) 182.

2. Observations by Seares † in 1908 showed the character of the variation and gave a visual range of about 0·85 mag., with the preliminary elements of maximum

J.D. 2418 132·805 + 0·44185 E.

From photographic observations at Dunsink a maximum is inferred at J.D. 2422 227·411, implying the lapse of 9267 periods in an interval of 4094·606 days. Hence the elements now adopted for maximum are:

J.D. 2422 227·411 + 0·441848 E,

which represent a very slight adjustment of the previous elements. Both sets rest on short series of observations, and there may be a doubt whether the number of periods in the interval has been rightly assumed. The period should therefore be checked later, but the present elements are good enough for the reduction of the observations.

3. Information relating to the comparison stars is given in Table I. The magnitudes m_1, m_2 are based on Schwarzschild's system for the χ Persei cluster, through two plates bearing duplicate exposures on the cluster and on the region of SW Andromedæ. The mean magnitudes m are adopted in the reduction of the variable, the plate residuals giving small mean

Table I.

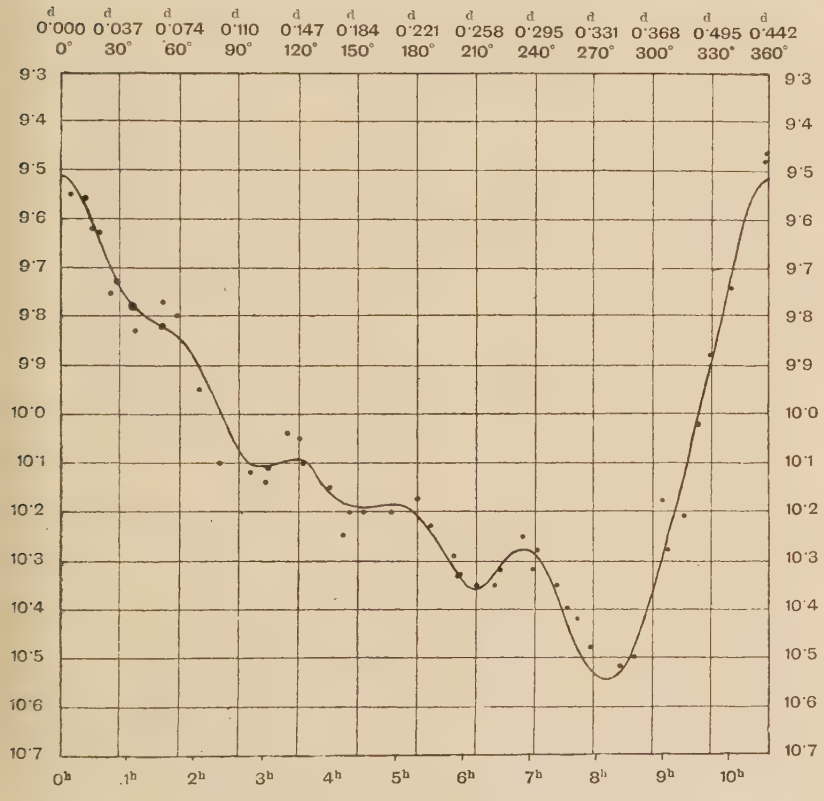
B.D.+28°.	Pl. 2097. m_1.	Pl. 2098. m_2.	Mean. m.	Corr.	Final. m_1.	P.E.	Range.
55	9·19	9·16	9·17	−·010	9·16	±·016	·13
58	9·54	9·46	9·50	+·009	9·51	±·015	·14
53	10·44	10·40	10·42	+·001	10·42	±·009	·07
59	10·51	10·46	10·49	·000	10·49	±·008	·06

corrections which are incorporated in m_1. From the same residuals, of which the extreme range is given in the last column, the probable error for a single plate is inferred and shown in the preceding column. They are naturally reduced by the fact that

* *H.C.O. Circular*, 124; *A.N.*, 4159, 4212. † *Laws Obs. Bulletin*, 16.

only four standards are suitably placed, but their size is evidence that these standards are sensibly free from variability.

4. The results from 47 exposures on the variable are given in Table II. In the second and third columns are recorded the astronomical date and G.M.T., together with the Julian day and decimal fraction, corresponding to the middle of each exposure, the length of which is shown in the fifth column. The phase in



Photographic Magnitude Curve of SW Andromedæ.

days, given in the fourth column, is based on the elements adopted in § 2. The curve from which the differences shown in the seventh column are derived has the equation

$$m = 10{\cdot}09 + 0{\cdot}33 \sin(\theta + 232^\circ) + 0{\cdot}19 \sin(2\theta + 255^\circ) + 0{\cdot}09 \sin(3\theta + 285^\circ) + 0{\cdot}05 \sin(4\theta + 346^\circ) + 0{\cdot}04 \sin(6\theta + 268^\circ),$$

where θ increases uniformly with the time and is reckoned from the adopted epoch of maximum, an increase of 2π corresponding to the period of the star. According to these residuals the

TABLE II.

Serial No.	Date G.M.T.	Julian Day.	Phase in Days.	Exposure.	Magn.	Obs. —Curve.	Images.
	1919. d h m	2422		m			
1	Aug. 29 11 44	200·490	·032	40	9·75	+·05	Good.
2	Sept. 2 12 49	204·534	·099	30	10·10	+·11	Good.
3	20 10 18	222·430	·321	30	10·42	−·05	Good.
4	20 11 35	222·483	·374	30	10·18	−·12	Fair.
5	25 9 49	227·409	·440	30	9·48	−·04	Fair.
6	25 10 21	227·431	·020	32	9·62	+·01	Good.
7	25 11 0	227·458	·047	30	9·83	+·05	Good.
8	25 11 23	227·474	·063	15	9·82	·00	Good.
9	27 10 12	229·425	·247	30	10·33	·00	Fair.
10	27 10 43	229·448	·270	30	10·35	+·02	Good.
11	27 11 19	229·472	·294	30	10·32	+·04	Fair.
12	27 11 50	229·493	·315	30	10·40	−·02	Good.
13	Oct. 2 11 53	234·495	·014	30	9·56	·00	Good.
14	2 12 24	234·517	·036	30	9·73	·00	Good.
15	2 13 4	234·545	·064	30	9·77	−·05	Good.
16	2 13 34	234·566	·085	30	9·95	+·05	Good.
17	14 7 48	246·326	·357	30	10·50	+·02	Fair.
18	14 8 20	246·347	·378	32	10·28	+·02	Fair.
19	14 8 58	246·374	·405	30	9·88	−·02	Good.
20	15 10 5	247·421	·127	30	10·14	+·03	Good.
21	15 10 36	247·442	·148	30	10·05	−·04	Good.
22	16 7 22	248·307	·129	30	10·11	+·01	Fair.
23	16 7 53	248·329	·151	32	10·10	+·01	Good.
24	16 8 29	248·354	·176	30	10·25	+·07	Good.
25	26 7 5	258·296	·397	30	10·02	+·02	Fair.
26	26 7 35	258·317	·418	30	9·74	+·02	Fair.
27	26 8 11	258·340	·441	30	9·46	−·06	Good.
28	26 8 43	258·364	·024	32	9·63	−·01	Good.
29	27 8 14	259·343	·119	30	10·12	+·02	Good.
30	27 8 46	259·365	·141	32	10·04	−·05	Fair.
31	27 9 23	259·391	·167	30	10·15	−·01	Good.
32	27 9 55	259·413	·189	32	10·20	+·01	Good.
33	27 10 42	259·447	·223	32	10·17	−·04	Good.
34	27 11 16	259·469	·245	34	10·29	−·03	Good.
35	27 11 56	259·497	·273	32	10·32	·00	Good.
36	27 12 30	259·521	·297	34	10·28	−·01	Good.
37	28 6 53	260·287	·179	29	10·20	+·01	Fair.
38	28 7 32	260·314	·206	34	10·20	+·01	Good.

TABLE II.—*continued.*

Serial No.	Date G.M.T.			Julian Day.	Phase in Days.	Exposure.	Magn.	Obs. —Curve	Images.
	1919.	d	h m	2422		m			
39	Oct.	28	8 6	260·337	·229	34	10·23	·00	Good.
40		28	8 49	260·367	·259	40	10·35	−·01	Good.
41		28	9 58	260·416	·308	40	10·35	−·01	Fair.
42	Nov.	11	7 39	274·320	·073	40	9·80	−·04	Fair.
43		13	7 14	276·302	·288	46	10·25	−·03	Good.
44		13	8 16	276·344	·330	60	10·48	−·05	Good.
45		25	7 2	288·293	·349	44	10·52	−·01	Good.
46		25	7 57	288·332	·388	44	10·21	+·07	Good.
47		26	6 38	289·277	·007	44	9·55	+·03	Good.

probable error of a measured magnitude of the variable is ±·024. There is no other harmonic term with a coefficient amounting to 0·02. By a change of epoch the equation can be written in the form

$$m = 10{\cdot}09 - 0{\cdot}33\cos\phi + 0{\cdot}19\cos(2\phi - 119^\circ) + 0{\cdot}09\cos(3\phi - 51^\circ) + \ldots$$

By the phases of the terms in this form SW Andromedæ is classed with RR Lyræ and SU Draconis, and with *o* Ceti. Its photographic magnitude curve, which is based on the above equation in θ and is shown in the figure on p. 213, resembles that of RR Lyræ in several respects.

Micrometrical Measures of 248 *Double Stars.* (List VII.)

By W. S. Franks.

The accompanying measures were made during the past twelve months with the 6½-inch Cooke equatorial of the Brockhurst Observatory, using the Cooke position micrometer and bright-field illumination as before. In the absence of a slipping-piece and bisecting screw—with which most modern micrometers are provided—all bisections were necessarily made with the slow-motion rods of the telescope.

Abbreviations.—Arg, *Argelander*; A.G., *Astronomische Gesellschaft*; *β*, *Burnham*; Cin, *Cincinnati Obs.*; "Cel. Obj.," Webb's *Celestial Objects*; Engel, *Engelmann*; Es, *Espin*; Fr, *Franks*; *h*, *Sir J. Herschel*; Hd, *Harvard*; Ll, *Lalande*; OΣΣ, *Poulkova Catalogue*, part ii.; S, *Sir J. South*; Schj, *Schjellerup*; Σ, W. Struve, *Mensuræ Micrometricæ*; Σ *rej*, *Catalogus novus*; Ang, *angle*; D, *distance*; <, *increasing*; >, *decreasing*; s, *slow or small*; p.m., *proper motion*; cols, *colours*; orge, *orange*; yel, *yellow*; ylsh, *yellowish*; grn, *green*; grnsh, *greenish*; bl, *blue*; blsh, *bluish*; prp, *purple*; prpsh, *purplish*; wh, *white*; p, *pale*; o, *no colour seen*; Quad, *quadruple*; Trip, *triple*.

In the absence of any remark, it is to be inferred that no change of position is indicated.

Measures of Double Stars.

Syn.	R.A. 1900.	Decl. 1900.	Magnitudes.	Pos. Ang.	Distance.	Epoch.	Nts.	Remarks.
	h m	° ′		°	″			
Σ 3065 *rej.*	0 2·9	−14 48	Fr 8·7, 8·9	288·8	9·64	1918·717	2	
,, 3064 ,,	,, 3·2	+39 32	,, 6·8, 9·8	358·6	23·72	·717	2	p. yel, bl.
h 1939	,, 4·7	+10 51	7·0, 10·0	161·5	35·37	·728	2	wh, o.
Weisse 1	,, 12·2	+35 16	Fr 8·4, 8·5	108·0	5·31	·728	2	Ang<s.
Σ 21 *rej.*	,, 12·2	+ 1 45	,, 9·2, 9·4	51·2	7·39	·728	2	
Es 116	,, 25·7	+55 5	9·0, 9·1	257·2	7·39	·728	2	wh, wh.
h 5451	,, 26·1	+33 2	6·0, 9·0	85·1	55·84	·728	2	yel, bl; BAC 120.
S 387	,, 29·6	+18 21	8·4, 8·4	233·0	42·11	·734	2	wh, wh.
Σ 43	,, 31·0	+59 58	8·5, 9·0	168·7	4·60	·728	2	wh, prpsh; Ang<s.
,, 56 *rej.*	,, 39·0	+33 0	9·0, 10·2	192·6	18·16	·734	2	=Σ 54; Ang>?
,, 58 ,,	,, 40·1	+ 9 45	Fr 8·0, 9·5	172·5	43·96	·734	2	
Es 43	,, 40·1	+54 26	8·0, 9·0	122·6	12·80AB	·738	2	yel, bl; Ang<D>.
Σ 77	,, 52·7	+26 22	9·1, 9·1	119·2	10·02	·738	2	
Winnecke 1	,, 52·8	+ 8 44	9·0, 9·2	128·7	5·44	·734	2	
S 390	,, 53·2	−16 13	Fr 7·0, 7·0	36·0	6·50	·717	2	wh, wh.
A.G. 11	,, 55·7	+11 51	,, 8·7, 8·8	4·3	52·58	·738	2	
Σ 110	1 12·9	−12 52	8·0, 8·5	355·7	7·82	·798	2	
,, 125	,, 21·9	− 0 40	Fr 8·5, 10·3	337·0	45·93	·785	2	Ang>D<; p.m.
h 3437	,, 23·3	−17 47	,, 8·0, 9·0	247·9	12·16	·798	2	
A.G. 19	,, 24·5	+13 41	,, 8·6, 8·9	322·3	101·13	·798	2	wh, wh.
Σ 128 *rej.*	,, 25·0	+60 31	,, 8·0, 9·7	305·3	12·91	·743	2	ylsh wh, blsh.
,, 153	,, 39·6	+60 46	,, 9·0, 9·7	69·8	7·85	·743	2	Σ 8·5; overrated.
A.G. 23	,, 40·4	+14 30	,, 9·2, 9·3	45·1	29·49	·807	2	
Σ 161 *rej.*	,, 41·9	+27 59	,, 8·5, 9·5	206·6	21·70	·786	2	
,, 162	,, 43·1	+47 24	7·0, 9·5	179·5	21·07AC	·786	2	very wh, bl.
Arg 6	,, 45·8	+56 48	Fr 8·0, 9·2	136·6	14·16	·812	2	BD+56° 372.
Σ 181 *rej.*	,, 48·7	+37 42	,, 8·7, 9·5	125·5	26·81	·812	2	
,, 196	,, 54·0	+20 31	8·5, 9·2	163·2	30·87AC	·812	2	yel, blsh; Ang and D>
,, 209 *rej.*	,, 58·1	− 7 53	Fr 8·8, 9·3	137·0	39·28	·807	2	
h 2104	,, 58·5	+52 33	,, 8·5, 8·7	167·0	30·32	·852	2	wh, ylsh.
Σ 210 *rej.*	,, 59·7	+36 28	,, 8·7, 9·6	243·0	17·08	·834	2	
Arg 7	2 3·8	+55 56	,, 8·8, 9·3	264·3	18·56	·852	2	BD+55° 531.
Σ 217 *rej.*	,, 3·8	+54 45	,, 8·8, 9·0	124·1	103·04AB	·851	2	} Triple.
,,	,, ,,	,, ,,	,, ,, ,,	9·0	128·2	134·39AC	·851	2
,, 224	,, 5·5	+13 12	7·5, 8·0	244·1	5·30	·858	2	ylsh wh, blsh wh.
A.G. 33	,, 5·8	+11 52	9·0, 9·6	320·4	40·40	·858	2	
Hd 57	,, 10·0	+23 58	9·2, 9·3	93·9	7·84	·858	2	
S 409	,, 11·0	+57 4	6·2, 9·0	136·1	121·85	·852	2	7 Persei; yel, bl.

Measures of Double Stars—continued.

Syn.	R.A. 1900.	Decl. 1900.	Magnitudes.	Pos. Ang.	Distance.	Epoch.	Nts.	Remarks.
	h m	° ′		°	″			
Σ 254	2 15·9	+23 11	8·5, 10·0	350·6	12·19	1918·885	2	
A.G. 38	,, 17·8	+14 58	8·7, 9·0	261·0	34·13	·887	2	wh, wh.
Σ 258	,, 17·9	+33 3	7·7, 8·9	147·0	70·35AC	·885	2	yel, blsh } Triple.
,,	,, ,,	,, ,,	8·9, 9·2	30·3	6·14CD	·885	2	blsh, ,, }
h 2137	,, 21·9	+42 47	8·6, 9·7	133·5	26·60	·888	2	
,, 3506	,, 29·5	−28 40 Fr	6·0, 8·5	244·5	10·46	·898	2	*ω* Fornacis; wh, blsh.
Σ 308	,, 42·1	−10 17	8·7, 9·2	334·1	21·03	·894	2	
h 3533	,, 45·2	−20 40 Fr	7·0, 7·5	273·7	39·42	·900	2	wh, wh.
Σ 326	,, 49·7	+26 28	7·5, 9·7	219·8	8·36	·894	2	yel, blsh.
,, 354 *rej.*	3 2·1	+24 11	8·5, 9·3	53·1	33·74	·927	2	
,, 345	,, 3·7	+78 7	8·0, 9·8	84·1	7·06	·909	2	p. yel, bl.
,, 373 *rej.*	,, 13·9	+62 22	7·0, 9·3	117·0	20·17AB	·909	2	ylsh wh, bl.
,, Trip.	,, ,,	,, ,,	,, 7·1	110·6	117·26AC	·909	2	ylsh wh, ylsh, wh.
h 3565	,, 14·1	−18 55	6·0, 9·0	116·3	6·30	·900	2	yel, bl.
Σ 376	,, 14·7	+19 22	8·0, 8·0	251·5	6·88	·947	2	Fr mags=; So 1884, 1910.
Schj 2	,, 20·3	− 1 31	8·0, 9·0	183·2	17·24	·947	2	
Σ 394	,, 22·3	+20 9	7·0, 8·0	161·9	6·85	·976	2	p. yel, bl.
,, 409 *rej.*	,, 26·7	+11 12	7·5, 9·5	252·7	17·03	·976	2	Ang<D>.
,, 415	,, 29·3	+26 31	8·3, 10·0	52·3	15·08	·976	2	
,, 446	,, 41·9	+52 21	7·0, 9·2	255·6	8·44	·976	2	wh, bl.
Σ 458	,, 43·9	+18 0	9·0, 9·3	201·0	5·12	·976	2	
,, 509	4 9·3	+61 40	7·7, 8·7	247·7	38·15AC	·998	2	wh, wh.
S 445	,, 13·4	+50 1	6·5, 7·2	327·4	74·05	·998	2	yel, p. bl.
h 2230	,, 20·1	+ 2 8 Fr	9·4, 9·5	325·5	34·26	1919·035	2	
Σ 560 *rej.*	,, 26·9	−13 52 ,,	6·5, 9·5	44·7	29·79	1918·998	2	wh, o.
,, 578	,, 34·9	+ 3 7 ,,	9·0, 9·5	25·0	11·43	·998	2	
Arg 11	,, 42·8	−17 26	8·4, 8·8	235·5	30·38	1919·018	2	wh, wh.
Σ 600 *rej.*	,, 45·3	+60 25 Fr	8·4, 10·4	52·0	30·50	·021	2	
Es 14	,, 53·7	+43 10	8·5, 9·0	157·1	32·98	·010	2	ylsh wh, blsh.
h 2244	,, 54·4	+69 14	8·0, 8·5	164·8	106·58	·018	2	yel, yel.
S 463	,, 56·2	+11 14	7·1, 9·4	28·9	31·96	·018	2	yel, bl.
,, 466	5 1·9	+21 35	6·5, 10·0	250·1	111·50	·052	2	105 Tauri; ylsh wh, o.
,, 470	,, 8·1	−17 34	8·0, 8·0	278·1	46·95	·035	2	wh, wh.
Es 59	,, 12·1	+33 25	8·5, 9·0	8·7	14·27	·052	2	
S 473	,, 13·2	−15 20	7·0, 8·5	306·9	20·35	·045	2	wh, blsh.
Webb	,, 17·9	+ 1 12 Fr	7·8, 10·0	12·5	64·34	·057	2	p. or, bl ("Cel. Obj.")
Σ 699	,, 18·8	+37 57	7·5, 8·0	343·5	8·70	·102	2	wh, blsh wh.
S 479	,, 19·5	+ 1 43 Fr	8·5, 8·7	220·0	46·40	·062	2	wh, wh.

Measures of Double Stars—continued.

Syn.	R.A. 1900. h m	Decl. 1900. ° ′	Magnitudes.	Pos. Ang. °	Distance. ″	Epoch.	Nts.	Remarks.
Σ 710	5 20·5	−11 24	8·2, 8·3	194·9	10·67	1919·102	2	wh, wh.
,, 711	,, 23·3	+54 35	7·5, 9·2	231·3	8·63	·102	2	yel, o.
Engel	,, 27·5	− 6 28	8·3, 9·0	251·5	44·61	·101	2	wh, wh.
A.G. 95	,, 27·6	+35 45 Fr	9·3, 9·7	16·3	24·74	·062	2	
S 490	,, 30·6	− 5 29 ,,	8·5, 9·5	214·6	78·25	·052	2	Involved in M42 neb.
,, 500	,, 43·2	+32 57	8·5, 8·5	89·3	59·94	·162	2	wh, wh.
Weisse 11	,, 47·4	+38 34	9·0, 9·0	164·9	24·65	·062	2	
h 374	,, 51·2	+27 22 Fr	9·2, 10·0	222·3	14·72	·164	2	
Σ 830	,, 57·1	+27 39	8·2, 8·7	253·1	11·97	·162	2	AB. yel, bl.
h 380	6 2·7	+34 30	9·0, 9·0	22·1	19·03	·170	2	
A.G. 106	,, 4·4	+33 4	9·0, 9·0	215·6	27·35	·170	2	
Σ 874 *rej.*	,, 7·6	− 3 38	8·7, 9·5	354·2	21·30	·170	2	
S 517	,, 17·9	−16 33 Fr	8·5, 8·5	192·3	22·81	·170	2	wh, wh.
Σ 930 *rej.*	,, 27·8	+ 8 5	8·5, 9·8	277·4	26·43	·201	2	AB } Triple.
,,	,, ,,	,, ,,	,,	9·5	105·6	43·15	·201	2 AC }
Σ 939	,, 30·6	+ 5 24	8·1, 8·7	105·8	29·98	·201	2	AB. yel, bl } Triple.
,,	,, ,,	,, ,,	,,	9·0	50·1	39·05	·201	2 AC. ,, ,, }
Σ 972 *rej.*	,, 43·8	−15 12 Fr	8·6, 9·4	35·9	10·00	·226	3	
Σ 1019	7 1·1	−10 30	7·0, 9·5	295·3	7·853	·220	2	wh, o.
,, 1026 *rej.*	,, 2·0	−11 9	6·2, 9·7	350·7	17·60	·220	2	ylsh wh, o.
,, 1047	,, 8·6	+15 57	7·3, 9·8	25·3	23·29	·232	2	ylsh, o ; Ang and D<.
,, 1064	,, 12·4	−11 51	7·0, 9·7	241·5	14·99	·220	2	ylsh wh, o ; Ang<s.
S 546	,, 14·4	+31 40	8·5, 8·8	358·5	84·08	·232	2	AB } Triple.
,,	,, ,,	,, ,,	,,	8·8	67·4	144·36	·232	2 AC }
Σ 1084	,, 19·0	− 3 47	7·2, 9·7	285·5	14·13	·226	3	yel, bl ; Lewis Ang wrongly reversed.
,, 1106	,, 25·6	+16 32	8·7, 8·7	32·8	10·87	·242	2	
,, 1128 *rej.*	,, 34·9	− 6 1	8·0, 10·0	169·0	16·19	·220	2	
,, 1153	,, 47·0	+12 17	9·0, 9·2	357·8	19·26	·242	2	
Weisse 16	,, 48·2	+41 50	9·1, 9·4	37·8	22·04	·245	2	
A.G. 147	8 0·4	+33 20 Fr	9·0, 9·5	140·3	11·30	·248	2	
Σ 1200	,, 8·6	+50 5	8·5, 8·5	360·0	8·15	·248	2	
,, 1210	,, 10·6	+ 3 7	7·2, 9·5	113·1	14·98	·248	2	wh, bl.
,, 1230	,, 22·8	+17 11	8·3, 10·0	189·0	31·00	·252	2	Ang>D<s.
,, 1238	,, 25·7	+33 30	8·0, 9·7	319·5	28·92	·252	2	
,, 1250	,, 32·9	+52 9 Fr	8·8, 9·0	167·5	21·55	·252	2	
S 584	,, 50·5	−11 0 ,,	7·0, 9·0	214·7	65·98	·248	2	Ang<D> ; wh, blsh.
Σ 1411 *rej.*	10 3·5	+32 51	9·0, 9·4	306·8	34·83	·297	2	
h 1176	,, 10·1	+58 7 Fr	9·2, 9·2	318·2	10·07	·297	2	

Measures of Double Stars—continued.

Syn.	R.A. 1900.	Decl. 1900.	Magnitudes.	Pos. Ang.	Distance.	Epoch.	Nts.	Remarks.
	h m							
Σ 1434	10 21·7	+18° 34′	8·5, 8·5	274°·0	6″·18	1919·297	2	
h 2531	,, 22·5	+40 43 Fr	9·0, 9·5	2·7	8·53	·297	2	
Σ 1446	,, 28·1	+15 45	8·5, 9·3	252·2	5·84	·297	2	
S 610	,, 29·1	−17 19 Fr	7·0, 7·3	43·7	105·68	·297	2	yel, yel ; Ang and D<.
Σ 1518 *rej.*	11 9·3	+ 5 49	7·5, 9·8	256·9	104·31	·349	2	AB ; wh, o.
Σ 3070	,, 19·4	− 3 50	8·9, 9·2	276·2	8·60	·349	2	
,, 1539	,, 23·0	+81 35	7·7, 9·2	310·5	19·54	·342	2	ylsh wh, blsh.
,, 1543	,, 23·7	+39 54	5·2, 8·2	357·9	5·52	·346	2	57 U. Maj. ; wh, purp ; Ang>.
h 500	,, 26·5	+36 25 Fr	9·0, 9·2	26·5	23·78	·355	2	
,, 513	,, 54·9	+26 37	8·5, 9·8	253·1	21·13	·360	2	
Küstner	,, 58·5	+33 38 Fr	8·6, 9·6	60·1	50·21	·360	2	BD +33° 2187.
,,	12 4·1	+32 29 ,,	8·7, 9·8	344·7	60·86	·375	2	,, +32° 2227.
Σ 1623 *rej.*	,, 11·1	+ 5 16 ,,	8·0, 9·4	182·4	45·90	·375	2	ylsh wh, blsh.
,, 1637 ,,	,, 18·6	+24 0 ,,	8·6, 9·7	143·9	121·60	·375	2	D<; p.m.
S 637	,, 22·0	−19 24	8·1, 8·6	202·3	60·26	·360	2	D>s.
Σ 1650	,, 26·6	+25 10	8·5, 10·0	179·1	16·50	·379	2	
,, 1662	,, 31·2	+57 8 Fr	7·7, 9·8	237·0	19·66	·379	2	yel, blsh ; Ang<s ?
,, 1688	,, 48·8	+38 38	8·5, 10·0	346·0	14·46	·379	2	
S 647	13 3·3	− 2 9 Fr	8·0, 9·5	211·3	41·15	·379	2	
,, 648	,, 8·3	+18 34 ,,	8·0, 8·5	66·7	86·79	·382	2	Ang<D>.
,, 650	,, 29·2	−12 55	8·0, 8·7	136·2	50·03	·379	2	,, >D<.
h 2663	,, 31·6	+20 30 Fr	8·5, 9·5	146·1	53·34	·383	3	*h* mgs underrated and Ang reversed.
S 652	,, 40·0	−10 1	8·0, 8·0	131·7	47·79	·382	2	Ang and D>.
,, 659	14 0·2	−17 36	7·3, 7·5	171·2	30·72	·382	2	ylsh wh, blsh wh.
h 2701	,, 3·6	+ 6 30 Fr	8·6, 8·8	36·0	12·54	·384	2	
,, 540	,, 5·3	+36 17 ,,	8·5, 8·8	212·2	8·62	·384	2	
S 665	,, 58·9	−17 31	7·7, 8·2	91·0	24·57	·382	2	wh, bl.
Σ 1918 *rej.*	15 6·0	+63 31	6·0, 10·0	21·2	17·98	·391	2	ylsh wh, o.
,, 1930	,, 14·2	+ 2 9	5·0, 9·8	36·8	11·13	·389	3	5 Serp. ; p. yel, purp.
Küstner	,, 24·2	+43 14	7·5, 9·5	318·5	41·04	·388	2	BD +43° 2500 ; p. yel, bl.
Σ 1955	,, 29·6	+27 3 Fr	8·7, 9·0	238·1	8·12	·388	2	
,, 1992	,, 55·6	−11 56 ,,	8·5, 8·7	324·0	6·34	·388	2	
,, 2009	16 0·3	+60 45	8·2, 10·0	304·1	17·73	·391	2	
,, 2040	,, 18·5	+14 4	8·0, 10·0	310·9	6·82	·402	2	
Arg (Es)	,, 26·9	+56 49	7·9, 9·0	58·7	77·56	·407	2	BD +56° 1892 ; wh, blsh.
Σ 2076	,, 35·6	+ 0 3 Fr	8·9, 9·8	324·8	9·51	·402	2	

Measures of Double Stars—continued.

Syn.	R.A. 1900.	Decl. 1900.	Magnitudes.	Pos. Ang.	Distance.	Epoch.	Nts.	Remarks.
	h m	° ′		°	″			
Küstner	16 35·8	+ 9 49	Fr 8·8, 9·8	47·7	57·12	1919·402	2	BD + 9° 3251.
Anon	,, 40·1	+ 6 16	,, 6·2, 9·8	191·4	163·06	·407	2	AB. 41 Herc (yel).
,, Trip.	,, ,,	,, ,,	,, ,, 10·0	246·5	157·98	·407	2	AC. P.m.
Σ 2134 *rej.*	17 2·8	+76 15	8·8, 9·4	146·3	38·62	·458	2	
A.G. 208	,, 3·1	+ 1 51	9·0, 9·2	242·5	27·36	·475	2	
Σ 2136	,, 8·2	+39 23	8·0, 10·0	113·6	14·94	·475	2	
,, 2143 *rej.*	,, 10·1	+10 6	Fr 8·4, 10·2	120·2	28·62	·475	2	
,, 2141 ,,	,, 11·6	+ 3 31	8·1, 9·5	125·5	33·02	·529	2	
,, 2167 ,,	,, 21·8	+49 37	8·0, 10·0	222·5	18·48	·456	2	
,, 2176	,, 26·4	+10 31	8·7, 9·7	11·5	16·78	·529	2	
,, 2189 *rej.*	,, 30·1	+47 57	7·9, 10·3	98·9	20·83	·456	2	AB.
,, Trip.	,, ,,	,, ,,	,, 8·6	359·9	64·86	·456	2	AC. ylsh wh, blsh wh.
Arg (Es)	,, 38·2	+59 11	Fr 9·0, 9·2	332·2	49·03	·456	2	BD + 59° 1833.
Σ 2223 *rej.*	,, 44·1	+ 5 1	,, 8·3, 9·5	209·9	18·28	·529	2	
OΣΣ 160	,, 48·8	+10 59	,, 8·2, 9·2	190·7	103·02	·529	2	OΣΣ missing pair (De), wh, blsh.
Σ 2279	18 2·0	+50 52	8·7, 8·8	180·2	13·01	·576	2	Ang > s?
,, 2300 *rej.*	,, 8·0	+59 44	Fr 8·5, 9·5	38·7	17·77	1918·762	2	AB.
,, Trip.	,, ,,	,, ,,	,, 10·3	87·5	24·66	·762	2	AD.
h 856	,, 9·4	− 4 43	8·7, 8·8	63·0	24·66	1919·538	2	wh, wh.
Küstner	,, 25·7	+29 55	Fr 9·0, 9·1	144·6	50·31	·576	2	BD + 29° 3272.
O. Stone 43	,, 30·9	+ 2 29	8·8, 10·0	30·2	10·29	·538	2	
Σ2345	,, 31·2	+21 0	8·4, 10·1	205·8	8·96	·576	2	Ang and D <.
,, 2366 *rej.*	,, 33·6	+69 52	8·2, 10·0	333·8	30·31	·595	2	D < ?
,, 2365 ,,	,, 34·5	+63 37	Fr 8·3, 9·5	22·8	19·94	1918·759	2	ylsh wh, bl.
S 704	,, 35·6	+ 9 36	8·2, 8·6	267·2	57·36	1919·538	2	wh, wh.
Es 787	,, 37·1	+51 1	Fr 8·7, 9·7	38·3	7·91	1918·762	2	
h 1339	,, 37·6	+46 0	,, 8·5, 10·0	326·5	24·60	·759	2	yel, o ; Ang >.
Σ 2394	,, 42·3	+41 56	8·7, 9·2	202·4	6·73	1919·595	2	ylsh, blsh.
,, 2399	,, 44·5	+13 6	8·2, 8·8	118·5	15·99	·547	2	AB. wh, blsh wh.
,, Trip.	,, ,,	,, ,,	,, 10·0	47·9	31·48	·547	2	AC.
Σ 2416	,, 49·7	+51 12	8·0, 10·2	155·6	17·44	·547	2	
,, 2428	,, 55·4	+14 46	8·0, 9·8	286·2	6·65	·547	2	P. XVIII 263 ; yel, bl.
,, 2432 *rej.*	,, 57·1	+12 24	7·5, 10·0	87·7	14·75	1918·748	2	ylsh wh, o.
,, 2444	,, 59·7	+25 54	8·5, 10·2	316·0	23·60	·759	2	Ang >.
S 710	19 1·1	−16 23	Fr 6·0, 9·0	5·7	7·09	·748	2	Ll 35693 ; wh, o.
Arg 33	,, 1·9	+57 19	8·2, 8·7	58·1	10·34	1919·595	2	ylsh wh, blsh wh.
Σ 2467	,, 4·2	+30 38	8·6, 9·0	264·5	10·30	1918·754	2	
h 881	,, 12·8	− 5 36	Fr 7·5, 9·5	341·1	32·85	·748	2	orge, blsh.
Σ 2538	,, 27·8	+36 30	8·2, 8·3	246·2	53·02	·754	2	AB.
,, Trip.	,, ,,	,, ,,	8·3, 8·7	53·8	6·09	·754	2	BC.

Measures of Double Stars—continued.

Syn.	R.A. 1900.	Decl. 1900.	Magnitudes.	Pos. Ang.	Distance.	Epoch.	Nts.	Remarks.
	h m	° ′		°	″			
Arg (Es)	19 28·1	+55 12	6·5, 9·0	86·3	76″30	1919·600	3	BD+55° 2211 ; p. yel, prpsh ; Ang & D<.
h 1414	,, 29·8	+35 58	8·7, 9·6	27·5	15·88	·616	2	
A.G. 232	,, 30·9	+35 5	8·8, 9·0	279·7	10·67	·616	2	
β 144	,, 33·9	+30 8	8·7, 8·8	351·5	6·86	1918·754	2	
Σ 2564	,, 36·0	+63 35 Fr	8·5, 9·5	167·3	9·49	·759	2	Ang and D>.
,, 2563	,, 38·0	+17 10	8·3, 9·5	285·0	6·04	·748	2	
,, 2577	,, 40·8	+20 39	8·1, 9·5	263·6	6·07	·754	2	yel, bl.
h 1439	,, 42·8	+55 36	7·0, 9·0	199·6	30·85	1919·601	2	wh, o.
β 147	,, 43·1	+31 51	8·2, 9·5	297·2	8·90	1918·754	2	yel, bl.
Σ 2588	,, 45·8	+44 8	7·9, 8·3	159·6	9·54	·754	2	ylsh wh, blsh, wh.
h 2906	,, 47·2	+59 1 Fr	9·3, 10·0	56·1	27·19	1919·605	3	Ang> ?
Σ 2590	,, 47·5	+10 5	7·1, 10·0	308·7	13·53	1918·748	2	P. XIX 307 ; ylsh wh, o.
Küstner	,, 49·1	+45 29 Fr	9·0, 9·3	265·8	53·77	1919·616	2	BD+45° 2993.
Σ 2602	,, 50·7	−13 37	8·5, 9·2	149·3	12·27	1918·748	2	
Arg (Es)	,, 50·8	+55 50	8·4, 9·2	307·1	56·67	1919·601	2	BD+55° 2275 ; D> ?
h 604	,, 52·3	+40 7	7·7, 9·3	93·8	70·26	·622	2	wh, o.
Hd 155	,, 52·5	− 9 20	7·7, 8·4	120·2	114·80	·600	2	AC. yel, blsh.
,, Trip.	,, ,,	,, ,,	,,	9·4	112·2	9·38	·600	2 AB. ,, bl.
Anon (*β*)	,, 53·7	+40 6	5·5, 9·0	317·3	64·66	·622	2	B.A.C. 6857 ; ylsh wh bl.
h 1457	,, 54·1	+37 39 Fr	9·3, 9·3	44·4	11·26	·628	2	
Küstner	,, 54·4	+31 31 ,,	9·0, 9·1	15·0	53 72	·622	2	BD+31° 3862.
h 1462	,, 55·9	+25 40	8·2, 10·0	24·3	34·07	1918·754	2	orge, bl ; Ang<s.
Σ 2615	,, 58·1	+ 8 7	7·2, 10·0	313·3	9·91	·748	2	wh, o ; Ang>.
Küstner	20 3·0	+ 9 59	9·0, 10·0	174·2	20·24	1919·636	2	BD+9° 4406.
Arg (Es)	,, 3·3	+57 50	9·0, 9·2	180·1	31·62	1918·765	2	BD+57° 2132-1.
S 737	,, 5·5	+20 42 Fr	7·7, 8·7	129·4	100·38	1919·636	2	ylsh, blsh.
h 2936	,, 5·7	+58 50 ,,	9·2, 9·4	253·5	12·74	·636	2	
,, 1492	,, 8·8	+28 54	9·2, 9·3	54·5	18·53	·650	2	
,, 1513	,, 20·2	+46 12 Fr	8·5, 9·5	327·3	13·72	·650	2	
,, 1523	,, 25·8	+40 40	8·8, 9·2	351·5	13·99	·650	2	Ang>.
,, 2974	,, 26·4	+19 47 Fr	9·2, 9·4	288·3	15·12	·665	3	
,, 1526	,, 26·7	+35 1	8·8, 9·0	148·2	8·85	·670	2	
,, 2977	,, 28·2	+17 42 Fr	9·5, 10·0	329·1	16·44	·665	3	
S 756	,, 28·2	+48 53	5·5, 10·0	323·2	57·40	·636	2	ω^3 Cygni ; orge, bl.
Σ 2697 *rej.*	,, 29·2	− 0 49 Fr	7·7, 9·9	2·0	29·38	·670	2	ylsh wh, o.
Schj 26	,, 29·8	+ 4 53 ,,	8·4, 9·6	88·5	24·36	·670	2	
h 609	,, 30·3	+40 13	8·7, 8·9	326·9	25·64	·685	2	
Arg (Es)	,, 55·1	+58 25	7·5, 9·0	197·2	55·07	1918·765	2	BD+58° 2195-4.
S 779	21 5·1	+38 19 Fr	8·0, 9·0	10·5	112·00	1919·685	2	ylsh wh, blsh wh.
Arg 42	,, 5·5	+47 51	8·2, 8·7	57·8	23·71	·688	2	

Measures of Double Stars—continued.

Syn.	R.A. 1900. h m	Decl. 1900. ° ′	Magnitudes.	Pos. Ang. °	Distance. ″	Epoch.	Nts.	Remarks.
Arg (Es)	21 9·4	+58 29	8·3, 9·0	193·7	36·57	1918·765	2	BD +58° 2237–6.
h 1629	,, 11·9	+46 13 Fr	9·0, 9·2	72·3	11·45	1919·688	2	
Es 98	,, 16·2	+51 54	,, 7·0, 9·0	310·7	26·70	·688	2	AB } ylsh wh, 2 blsh.
,, Trip.	,, ,,	,, ,,	,, ,,	9·2	87·6	30·09	·688	2 AC }
h 1645	,, 21·4	+49 48	8·5, 9·5	39·2	7·93	·688	2	
,, 1676	,, 35·8	+46 44	8·4, 9·7	136·0	28·82	·694	2	ylsh, blsh.
Küstner	,, 45·4	+33 47 Fr	9·0, 9·2	180·5	47·00	·694	2	BD +33° 4359.
Σ 2833	,, 47·0	+ 8 37	7·2, 10·0	334·4	9·14	1918·690	2	p. yel, blsh.
,, 2852	,, 57·0	+53 41	9·0, 9·1	172·5	7·71	·700	2	
h 1714	,, 57·3	+45 52 Fr	8·8, 8·8	251·3	16·80	1919·694	2	wh, wh.
,, 1735	22 5·1	+44 21	7·2, 8·7	109·7	26·81	·713	2	ylsh wh, bl.
,, 1741	,, 7·3	+50 20 Fr	6·2, 10·0	303·4	28·20	·727	2	wh, o; Ang>D<.
,, 965	,, 24·4	+34 1	8·4, 9·2	141·2	29·24	·713	2	ylsh wh, blsh.
Σ 2916	,, 26·9	+40 42	7·3, 8·8	336·4	44·83	1918·706	2	AB. yel, bl. Trip.
h 1785	,, 30·0	+29 12 Fr	8·5, 9·0	171·9	14·23	1919·713	2	
,, 5355	,, 33·0	−14 36	8·0, 8·2	293·9	77·28	·740	2	AB. wh, wh.
,, Trip.	,, ,,	,, ,,	,,	8·8	2·3	107·10	·740	2 AC. ,, ruddy.
,, 1813	,, 44·0	+41 4	8·9, 9·0	60·9	9·25	·740	2	
,, 1832	,, 53·1	+38 8	8·8, 9·2	72·0	11·43	·743	2	
S 821	,, 54·2	+27 12	8·2, 9·8	4·8	6·31	1918·706	2	=Σ 2967; Ang>s.
h 5384	23 1·4	−15 25 Fr	8·5, 10·0	194·9	29·55	·700	2	
Σ 2975 *rej.*	,, 1·9	+32 30	9·2, 9·2	291·0	31·81	1919·743	2	Ang and D<.
h 983	,, 9·6	+31 14	8·8, 9·5	157·5	16·64	·743	2	
Anon (Webb)	,, 33·6	+44 8	8·4, 8·5	305·6	125·38	·743	2	AB. BD +43° 4515–14.
Quadruple	"Curious group nearly an equilateral rhombus."—Webb.		,, 9·3	248·5	117·20	,,	,,	AC. ,, ,, −13.
,,			8·5, 9·4	252·7	104·38	,,	,,	BD. ,, 4514–10.
,,	Rather striking geometrical figure.—Fr.		,, 9·3	182·8	114·21	,,	,,	BC. ,, ,, −13.
,,			9·3, 9·4	311·7	129·18	,,	,,	CD. ,, 4513–10.
h 990	23 34·3	− 5 13	8·5, 10·2	284·7	41·75	1918·700	2	} Curious similarity.
Anon (Cin)	,, 34·6	− 4 53	8·5, 10·0	67·7	41·63	·683	2	
h 5413	,, 36·5	−18 22	5·0, 7·0	5·5	122·04	1919·750	2	104 Aquarii; p. yel, bl.
Küstner	,, 37·0	+14 50 Fr	9·0, 9·3	99·0	26·81	·752	2	BD +14° 5036.
Σ 3037	,, 41·3	+59 55	7·0, 8·8	187·5	29·27	1918·706	2	AC. Ang<s.
,, Quad.	,, ,,	,, ,,	,,	8·9	227·8	50·49	·706	2 AD. very yel, 2 blsh.
h (Cape Obs.)	,, 49·2	−27 36 Fr	6·3, 6·8	268·0	7·12	·743	2	ylsh wh, blsh wh.
h 5435	,, 52·2	−16 39	8·5, 9·0	8·1	14·97	·717	2	Ang<.
,, 1927	,, 58·0	+44 35	8·6, 8·9	75·5	15·28	1919·752	2	Ang>s.

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Measures of Double and Proper-Motion Stars. By Dr. J. E. de Vos van Steenwijk, F.R.A.S.

The following measures have been executed with the 10½-inch refractor at the Leyden Observatory, from May 1918–June 1919.

The telescope and micrometer have been described at length by J. Voûte, who used them for the same purpose, in part x. of the *Annalen der Sternwarte zu Leiden.* The revolution value of the micrometer-screw I have determined anew on several nights, by measurements of the AZ arc in Perseus and by chronographic transits. The value adopted was 30″·86, whereas Voûte used 30″·88.

The methods of observation did not differ in any respect from the usual ones, except that throughout the work wire-illumination was used; this was found very satisfactory for the faint and wide pairs, which formed the greater part of the programme, and did not lead to inconvenience, as the illumination is strictly symmetrical. The power used varied, according to the condition of the air and the required field of view, between 238 and 600.

The observing list was constructed with regard to the fact that the bad seeing prevailing at Leyden prevents the measuring of close pairs on many nights. Therefore a selection was made from Burnham's "Measures of proper-motion stars," and the list completed with pairs of the 61 Cygni type and others where relative motion was only surmised.

New relative proper motions have been derived for many Burnham pairs, especially when more than one set of older measures was at hand. For the comparison with meridian proper motions the reader is referred in most cases to the work cited above.

About those proper motions I want to say a few words. Though at present nobody can agree with the verdict of Burnham that a star below, say, the ninth magnitude, having a sensible proper motion is so improbable a thing that the relative motions found in such cases may be always accepted as wholly belonging to the brighter star, still I believe that the results deduced in this way retain their value. They may serve a double purpose. Firstly, as a good approximation in the case of those rapidly moving stars of which older meridian observations are wanting. *Individual* proper motions of rapidly moving stars (especially of the fainter ones) are of interest because they form the exceptions to the statistical rules of velocity distribution and belong probably to our nearer neighbours.

On the other hand, if a good meridian proper motion of the principal star is to be obtained, we can deduce the absolute proper motion of the companion. These form a valuable check on our rapidly accumulating stock of photographic proper motions. Also they might be directly used as Comstock did in his determination of the solar apex from very faint stars, which determination could now be repeated from much better data.

Explanation of the Table.

Most of the columns are self-explanatory. The nomenclature of Burnham is always followed even in the case of stars not given in the General Catalogue and in the lettering of multiple pairs. The magnitudes are also borrowed from Burnham in nearly all cases. If in the column giving the number of nights (n) a figure between brackets is added, this means that only on those nights complete measures have been taken; on the remaining nights only a position-angle has been measured. Under the heading remarks, (P.M.) means that the pair is to be found in Burnham's Measures of proper-motion stars and (De) in the Dearborn Catalogue by Fox. An asterisk refers to the more explicit notes at the end of the table. For rapidly moving pairs the yearly results are printed separately.

Name.	Epoch. 1900+	Mag.	P.A.	D.	n.	Remarks.
β 12740 AD	18·86	8·6, 11·0	179°·57	84″·71	1	(P.M.)
,, AE	18·86	8·6, 11·0	58·37	80·56	1	(P.M.)
Groomb. 34	18·76	8·3, 9·5	56·44	38·56	1	(P.M.)
β 190	18·69	8·8, 8·8	335·2	...	2	Twilight; no distance. (De.)
β 246	18·76	8·5, 9·0	113·2	5·55	3	Change in distance.
β 319	18·76	8·2, 9·5	11·46	62·37	1	(P.M.)
B.D. {32°·120 / 33°· 99	18·86	8·5, 9·5	180·34	283·5	1	(P.M.)
β 391	18·88	5·7, 9·7	160·17	35·84	2	*
β 426	18·94	4·0, 7·6	255·9	7·08	3	η Cassiop. (De.)
Lal. 1532	18·96	7·5, 9·8	215·24	60·16	1	(P.M.)
β 609	19·15	7·5, 8·0	191·15	14·62	3 (2)	One distance measure rejected.
β 697	18·96	6·6, 8·5	346·85	53·85	4 (3)	(P.M.) Change in angle now certain.
Lal. 2682 AB	18·85	8·2, 9·1	74·60	82·69	2	(P.M.) *
,, AD	18·85	8·2, 10·3	188·78	80·39	2 (1)	(P.M.) Epoch D = 18·94. *
Lal. 2701	18·71	8·5, 9·0	347·68	184·97	1	(P.M.)
B.D. 30° {229 / 230	18·71	8·7, 9·2	251·70	89·77	1	(P.M.)
β 860	18·90	8·2, 8·4	359·42	13·20	1	(P.M.)
β 878	18·81	9·2, 9·3	241·88	18·29	2	Relative proper motion but measures discordant. *
β 900	18·87	8·4, 8·7	234·8	...	1	
β 1008 Bb	18·91	6·0, 9·2	257·65	203·85	2	(P.M.)
β 1027	18·88	7·0, 8·5	24·5	...	1	Angle unchanged since 1900.
β 1131	18·77	7·0, 8·0	331·73	66·86	2	(P.M.)
B.D. 67°·191	18·72	7·8, 9·3	101·49	36·18	2	(P.M.)

Name.	Epoch. 1900+	Mag.	P.A.	D.	*n.*	Remarks.
			°	″		
β 1350	18·95	9·4, 9·4	62·0	3·83	2 (1)	Change in angle probable; distance measure unreliable.
β 1386	18·86	6·0, 9·2	313·4	...	1	
β 1393	18·82	4·0, 10·0	301·00	17·58	2	9 Persei.
β 1427	18·88	7·3, 8·2	315·0	3·56	2	
β 1520	18·94	9·4, 10·0	59·2	3·78	2	(De.) *
Lal. 5586	18·81	7·1, 9·2	265·92	85·90	2	(P.M.) DM. 35° {615 / 616 *
Lal. 5712	18·81	7·0, 8·4	246·62	119·34	2	(P.M.) Distance measures discordant.
β 1787	18·98	6·0, 8·2	253·0	6·50	2	
Br. 536	19·02	7·0, 9·5	153·80	95·82	3	(P.M.) *
39 Tauri	18·91	6·5, 8·2	1·98	170·09	2	(P.M.) *
β 2115	18·99	8·0, 8·0	118·6	...	1	(De.)
51 Tauri	18·85	5·9, 9·5	62·25	166·21	2 (1)	(P.M.)
β 2143	19·06	9·6, 9·6	203·4	3·70	1	(P.M.) Perhaps increase in distance.
β 2163	19·00	6·0, 8·8	35·0	4·72	3 (2)	(P.M.)
81 Tauri	19·05	5·9, 9·0	338·68	161·95	1	(P.M.) No change.
β 2232	19·04	9·5, 9·5	286·0	4·83	2	
β 2272	18·87	8·5, 9·0	325·2	2·10	1	(De.)
β 2284	18·88	6·5, 6·5	197·6	3·68	2	(De.)
β 2336	18·97	8·0, 8·0	294·4	4·34	2	
7 Orionis	19·08	5·0, 8·7	249·18	171·36	2	Distance measures discordant. (P.M.)
β 2435	18·88	4·0, 7·9	356·4	5·30	2	Change in angle and distance now certain. (P.M.)
β 2445	18·86	6·0, 9·7	259·9	3·18	1	
β 2525	19·05	9·5, 10·2	210·9	...	1	(De.)
β 3004	19·12	7·0, 9·0	111·3	...	1	(De.)
β 3074 AB	18·97	3·0, 7·5	339·6	2·64	2	(9 Aurigæ.)
,, AD	19·06	3·0, 9·5	349·60	130·29	2	(P.M.)
β 3423 AB	19·04	6·7, 7·7	330·0	4·76	2	Groomb. 1173; perhaps increase in angle, no change in distance.
,, BC	19·04	7·7, 10·0	27·02	136·60	2	*
β 3692	19·00	5·4, 7·7	157·8	6·58	2	
B.D. 22°·1620	19·15	7·2, 10·0	264·11	95·11	2	No change.
β 3949	19·18	7·5, 7·5	105·8	1·86	2	(De.)
β 4159	19·00	7·0, 7·0	178·01	64·27	2	*

Name.	Epoch. 1900+	Mag.	P.A.	D.	n.	Remarks.
			°	″		
β 4122	AB 19·00	2·0, 2·8	215·7	4·99	2	Castor.
,,	AC 19·18	2·0, 10·0	164·17	73·17	2	Comes estimated much brighter.
Lal. 15394	AB 19·11	7·9, 10·1	258·49	87·01	2	(P.M.)
,,	AC 19·11	7·9, 10·4	83·11	122·12	2	(P.M.)
β 4361	AB 19·19	5·5, 9·0	76·46	88·02	3 (2)	One distance measure (88″·79) rejected.
,,	AC 19·15	5·5, 9·1	150·60	119·95	2	
β 4402	19·24	7·8, 9·7	231·3	1·73	2	
β 4477	AC 19·09	5·0, 5·5	110·6	6·21	3 (2)	ζ Cancri.
Lal. 15976	AB 19·14	7·5, 9·0	159·82	131·36	3 (2)	Epoch dist. meas. 19·21. (P.M.) *
,,	BC 19·24	9·0, 11·0	153·07	101·32	2	(P.M.) Distance measures discordant. *
Lal. 16494	19·16	7·5, 9·5	43·79	155·51	1	(P.M.)
β 4609	AC 19·21	3·4, 10·0	150·46	142·92	2	(P.M.) *
β 4710	AC 19·26	6·5, 9·0	342·23	63·10	2	(P.M.)
,,	AD 19·26	6·5, 9·0	43·32	82·64	2	(P.M.)
39 & 40 Cancri	AB 19·09	6·7, 6·9	151·01	150·14	2	(P.M.)
,,	Aa 19·15	6·7, 9·0	309·10	134·42	2	(P.M.) Distance measures discordant.
,,	Ab 19·15	6·7, 9·0	110·68	135·43	2	(P.M.)
,,	BC 19·25	6·9, 10·8	147·03	140·03	1	(P.M.)
γ Cancri	19·21	5·0, 9·2	66·09	106·28	2	(P.M.) *
β 4771	19·29	3·8, 7·8	245·6	3·46	2	ε Hydræ.
Pi. VIII. 174	AB 19·15	8·0, 9·3	322·22	166·82	2	(P.M.) *
,,	AC 19·15	8·0, 9·4	281·06	222·88	2 (1)	One distance measure rejected. *
β 4815	AB 19·21	7·5, 7·6	48·8	4·38	2	(P.M.)
β 4819	19·18	7·0, 8·0	313·31	44·28	2	B = B.D. 26°·1854. *
β 4923	AC 19·21	5·0, 9·4	147·85	204·57	2 (1)	σ^2 Ursæ Maj. One dist. meas. rejected.
80 Cancri	AB 19·19	6·5, 9·1	252·01	133·36	2	(P.M.)
,,	BC 19·27	9·1, 9·6	317·30	148·32	1	(P.M.) No relative change since 1909.
β 4972	AB 19·18	7·4, 7·4	69·25	18·88	2	
,,	AC 19·27	7·4, 10·0	196·70	144·75	1	
β 4999	19·19	8·3, 8·5	249·90	18·08	2	
β 5023	AB 19·22	6·1, 10·2	126·3	5·56	1	Angle increasing, no change in distance.
,,	AC 19·16	6·1, 10·0	4·99	142·89	1	(P.M.)

Name.	Epoch. 1900+	Mag.	P.A.	D.	*n.*	Remarks.
			°	″		
β 5131	19·24	5·0, 9·5	74·72	37·23	2	(P.M.)
β 5171	19·21	7·0, 8·3	290·2	3·14	3 (2)	Groomb. 1569.
β 5175	19·26	4·0, 9·3	43·00	85·34	2	(P.M.).
Lal. 19627	19·21	8·8, 9·4	255·63	180·24	2	(P.M.) *
Groomb. 1618 AB	18·34	6·7, 8·8	187·12	165·44	2	(P.M.) *
β 5356 AB	18·76	6·1, 7·0	167·42	16·64	2	Fixed.
,, AC	19·19	6·1, 10·0	11·50	150·06	1	(P.M.)
Groomb. 1636	18·76	6·5, 8·8	102·60	150·58	2	(P.M.) *
β 5385	19·25	8·6, 9·3	50·9	...	2	
β 5695 AB	18·36	7·7, 8·7	265·60	35·15	2	*
,, AC	18·70	7·7, 8·0	334·84	167·56	3	(P.M.)
β 5721	19·00	7·4, 8·0	257·29	80·35	4	(P.M.)
β 5734	18·37	4·4, 4·9	110·8	3·20	4	ξ Ursæ Majoris.
	19·22	4·4, 4·9	105·9	3·09	1	
β 5739	19·29	6·9, 8·1	17·8	3·17	4	Distance decreasing.
β 5850	19·32	9·5, 9·8	29·2	2·82	3 (2)	Change in angle confirmed, *cf.* De.
β 5858 AB	19·16	6·0, 8·0	258·36	9·99	3	
,, AC	19·25	6·0, 10·0	88·81	114·41	3	(P.M.) Comes estimated much brighter.
β 5887 AC	18·34	9·3, 9·2	275·83	45·06	2	Measures uncertain.
β 6043	19·24	9·7, 9·9	258·07	9·40	3 (2)	De. *
Lal. { 22662 22667 }	18·68	8·8, 9·0	97·59	113·71	3	(P.M.) *
β 6064	18·61	7·5, 7·7	221·85	12·27	4	(P.M.) Increase of distance confirmed; angle perhaps decreasing.
β 6084 AC	19·31	7·5, 9·7	293·18	156·84	3	(P-M.) *
,, AD	19·38	7·5, 9·0	Δδ = 136″19	...	3	(P.M.) *
β 6090 AB	18·90	7·5, 7·8	279·2	7·24	3	(P.M.)
,, AC	18·90	...	176·34	98·72	3	(P.M.)
β 6154	19·32	9·3, 9·6	148·00	23·86	3 (2)	(P.M.) Change in distance not confirmed.
7 Can. Ven. AC	19·00	6·5, 8·7	327·24	229·01	5 (3)	(P.M.)
χ Virginis AB	19·04	5·0, 9·0	137·22	173·44	3	(P.M.) *
,, AC	19·38	5·0, 10·3	110·25	...	2	(P.M.) Through clouds.
β 6230	19·01	8·0, 9·3	107·71	53·47	3	Distance measures erratic. (P.M.)
β 6242	18·85	6·5, 9·2	281·45	84·71	4 (3)	Epoch D = 19·01; no relative change.

Name.		Epoch. 1900+	Mag.	P.A.	D.	*n.*	Remarks.
34 Virginis		18·99	6·2, 9·4	° 3·82	″ 139·27	3	(P.M.) *
Pi. XII. 188	AB	19·20	7·0, 10·0	355·63	158·73	3 (2)	(P.M.) *
,,	BC	18·95	10·0, 9·4	328·10	135·58	3 (2)	(P.M.) *
β 6289	AB	18·35	6·8, 7·3	202·07	16·30	4	
,,	AC	18·68	6·8, 8·0	325·62	246·04	3	(P.M.)
,,	BC	18·37	...	328·7	255·94	1	(P.M.)
β 6318	AB	18·70	6·3, 8·2	284·0	3·91	4	(De.) Common prop. motion, and perhaps increase in distance.
β 6389		19·14	6·5, 8·5	254·0	...	3	Change in angle confirmed, *cf.* Gen. Cat.
β 6405	AB	18·86	4·0, 9·0	343·1	7·02	2	Fixed.
,,	AC	19·09	4·0, 9·0	299·22	71·61	4	(P.M.)
β 6476		18·91	8·3, 8·5	345·5	...	4	Angle still increasing. Difficult object.
β 6566	AC	18·36	5·0, 10·5	123·0	1·42	2	
,,	AD	18·40	5·0, 9·5	140·40	220·25	2	
β 6586		18·86	6·2, 9·1	141·1	4·66	4 (1)	Epoch D = 19·35.
β 6776		18·40	8·2, 8·5	80·1	...	2	
		19·35	8·2, 8·5	84·6	2·25	2	
β 6778		18·87	5·1, 7·2	236·76	13·10	4	*
β 6837		18·88	7·0, 7·0	171·8	5·73	4	Change in distance confirmed. *
β 6869	AB	18·36	9·0, 9·0	74·90	45·60	2	(P.M.)
,,	AC	18·36	9·0, 8·9	117·40	163·61	2	(P.M.)
		19·37	9·0, 8·9	117·27	162·48	2	Very large proper motion.
9 Boötis		18·36	4·4, 11·0	181·49	69·11	1	(P.M.) No change.
β 7040		19·07	8·5, 10·2	201·8	...	3	
45 Boötis	AC	18·34	5·2, 9·5	41·10	244·14	2	(P.M.)
,,	,,	19·39	5·2, 9·5	40·74	243·97	2	
o Coronæ		18·89	5·7, 9·6	337·98	146·96	4	(P.M.) Older measures discordant.
β 7222		18·91	4·7, 9·3	16·6	2·93	4	
β 7259		18·36	7·3, 7·9	50·5	1·79	2	μ_2 Boötis. *
,,	AC	18·36	7·3, 4·9	171·25	108·55	2	$\mu_1 - \mu_2$ Boötis.
β 7277	BC	19·14	9·5, 9·5	282·1	8·81	4	
,,	AE	19·36	8·8, 10·7	53·27	146·48	2	
β 7327		18·89	8·5, 8·5	304·73	56·09	4	(P.M.)

Name.		Epoch. 1900+	Mag.	P.A.	D.	*n.*	Remarks.
β 7332	AB	18·36	6·8, 7·9	199°·4	1″·08	2	
,,	AC	18·63	6·8, 7·0	327·91	122·22	4	
ψ Serpentis	AB	18·90	6·2, 9·1	207·65	207·26	4	(P.M.)
,,	BC	18·90	9·1, 7·4	235·56	172·08	4	(P.M.)
,,	*Aa*	18·73	6·2, 10·8	281·20	171·54	3 (2)	(P.M.) Epoch D = 18·42. *
β 7442		18·92	5·7, 9·5	64·71	96·30	4	*
15 Herculis	AB	19·15	7·5, 9·8	257·40	65·07	4	(P.M.) *
,,	BC	18·43	9·8, 11·0	17·72	108·37	1	(P.M.)
,,	AC	19·36	...	341·06	109·49	2 (1)	
β 7466	AB	18·38	8·2, 8·2	39·68	27·72	2	
,,	AC	18·75	8·2, 9·5	178·60	242·15	3 (2)	Epoch D = 18·93.
β 7500	AB	18·90	6·5, 8·0	325·32	35·27	4	
,,	AC	18·90	6·5, 9·8	137·57	167·34	4	
β 7563	AB	18·94	4·9, 5·2	218·1	5·18	2	
,,	AD	18·97	4·9, 10·3	85·46	66·52	2	Distance measures discordant.
β 7649	AB	18·38	4·0, 6·1	84·6	1·05	1	λ Ophiuchi.
β 7649	*Aa*	18·41	4·0, 10·0	246·63	313·60	1	No older measures.
,,	AC	18·92	4·0, 8·8	169·65	119·14	2 (1)	(P.M.) Epoch D = 18·44.
β 7699		19·12	8·0, 9·5	237·80	4·68	4 (3)	Epoch D = 19·03. *
30 Ophiuchi		19·08	5·0, 9·8	69·33	94·12	3	(P.M.) *
β 7792		19·24	7·0, 10·3	185·2	...	4	Kü. 1. (De.) decrease in angle.
β 7845	AB	18·97	5·8, 6·3	116·00	298·90	4	
,,	*Ba*	18·97	6·3, 9·0	267·98	152·35	4	Distance measures very discordant.
,,	*ab*	19·04	9·0, 9·5	209·58	98·71	4	Some relative motion.
β 7858	AC	18·85	6·4, 9·6	174·20	146·04	3	(P.M.) *
β 7863	AB–C	19·06	8·2, 9·2	85·22	172·33	3	(P.M.) C = B.D. 0°·3634.
B.D. 42°	2810	18·43	9·4, 9·2	280·44	204·42	2 (1)	(P.M.) *
,,	2808	19·41	9·4, 9·2	280·57	202·90	1	(P.M.)
β 7922		18·98	3·0, 8·1	200·54	12·41	4	δ Herculis.
β 8068	AC	19·18	7·0, 7·7	211·70	82·64	4	
,,	CD	18·44	7·7, 10·5	180·32	143·75	1	(P.M.)
B.D. 18°·3424	AB	18·48	9·0, 10·3	265·00	267·86	1	(Through clouds.) Δδ = 23″·36.
,,		19·40	9·0, 10·3	only Δδ	23·05	2	(P.M.) *
,,	AC	19·38	9·0, 9·0	,, Δδ	5·46	3	*
β 8162	A–BC	18·55	10·0, 3·4	245·68	32·81	2	μ Herculis.

Name.		Epoch. 1900 +	Mag.		P.A.	D.	*n.*	Remarks.
					°	″		
β 8211		19·10	7·5,	9·1	124·80	19·01	3	(P.M.) Increase in distance not confirmed.
β 8295	AB	19·24	7·0,	7·1	41·61	57·21	4	*
,,	BC	19·83	7·1,	9·8	339·27	184·74	3	Epoch D = 19·94. *
β 8354	AD	18·98	6·8,	9·3	191·12	200·74	2	
Pi. XVII. 368		18·62	7·7,	10·0	240·92	131·89	1	(P.M.)
β 8404		18·77	8·5,	9·0	339·92	27·57	4 (3)	Epoch D = 18·86. *
β 8574	AB	19·00	4·7,	7·7	267·4	3·85	4	Increase in distance confirmed.
,,	AC	18·82	4·7,	7·1	20·32	89·09	3	
β 8894	AC	18·57	7·0,	8·5	5·22	150·70	2	
,,	BD	18·57	7·1,	9·0	358·80	149·51	2	
β 8986		19·02	7·9,	8·4	118·65	23·92	4	
Lal. 35851		18·95	5·5,	9·5	276·69	122·21	3 (2)	Epoch D = 18·65. Distance uncertain.
Oxf. 26°·55357		18·87	8·4,	9·5	38·47	80·39	3	B.D. 25° {3780 3783 *
β 9137		18·80	6·0,	6·5	217·0	8·96	4	
β 9150		19·01	8·5,	9·1	271·2	3·29	4	(De.) Holden 34. *
β 9308	AC	18·65	5·5,	9·5	322·74	70·72	2	
,,	AD	18·64	5·5,	9·5	63·70	149·62	2 (1)	One distance measure rejected.
β 9493		18·58	9·2,	9·4	190·0	7·94	2	(P.M.) Decrease in distance not confirmed ; fixed.
β 9560		18·79	5·1,	5·3	134·58	38·64	4	
β 9607	AD	18·68	6·6,	9·5	250·99	143·52	3 (2)	(P.M.) Change in both co-ordinates confirmed. *
β 9617		18·93	5·1,	8·1	70·42	25·75	3	Change still doubtful.
β 9652		18·43	9·1,	9·2	274·9	...	1	*
β 9713		18·92	4·0,	7·6	8·2	3·24	4 (2)	Epoch D = 19·22. (ε Draconis.)
Lal. 37626		19·10	7·2,	9·5	119·72	100·77	2 (1)	Epoch D = 18·68. (P.M.)
15 Sagittæ	AB	18·66	6·0,	8·8	275·93	190·73	2	(P.M.) *
,,	AC	18·66	6·0,	7·0	319·95	203·70	2	(P.M.) *
,,	*Bb*	18·66	8·8,	9·2	230·92	183·44	2	(P.M.) *
β 9949		18·96	8·0,	9·5	13·0	5·28	3	(P.M.)
29 Cygni	AB	18·77	5·0,	7·0	153·37	212·65	2	(P.M.) *
,,	AC	18·71	5·0,	10·0	22·89	221·71	2	*
,,	BD	18·71	7·0,	10·6	119·99	217·40	2	*

Name.		Epoch. 1900+	Mag.	P.A.	D.	*n.*	Remarks.
β 10044		18·93	7·0, 9·1	115°·6	5″·37	3	(P.M.)
Cephei H 6		18·94	4·7, 9·5	161·39	66·42	3	B=B.D. 57°·2241. *
Lal. 40604	AC	18·80	7·8, 9·0	23·59	152·92	2	(P.M.) *
,,	BC	18·66	10·0, 9·0	69·30	87·60	2	(P.M.) *
,,	CD	18·66	9·0, 10·2	356·10	92·84	2	(P.M.)
β 10587		18·69	8·7, 10·2	304·6	9·98	2	(P.M.) Relative motion confirmed.
61 Cygni	AB	18·69	5·6, 6·3	131·75	23·95	3	β 10732.
,,	AC	18·66	5·6, 10·5	31·07	157·13	2 (1)	No older measures.
,,	AD	18·82	5·6, 8·5	282·25	304·89	2	,,
61 Cygni	AE	18·82	5·6, 8·0	12·28	305·97	2	No older measures.
B.D. 10°·4481		18·80	8·6, 9·5	356·87	186·45	3	(P.M.) *
Oxf. 26°·74272		18·66	9·4, 10·3	19·70	126·64	2	(P.M.) *
β 11214	AD	18·79	4·0, 7·0	52·90	204·61	2	(μ Cygni.)
β 11355		18·67	9·2, 9·3	164·25	2·53	3 (1)	The single distance measure probably erroneous.
β 11483		19·14	4·7, 6·5	280·5	7·19	4	(ξ Cephei.)
Br. 2926		18·64	5·4, 10·5	33·57	72·88	2	(P.M.) * Companion estimated much brighter.
ε Cephei		18·64	4·5, 9·6	15·40	127·80	2	(P.M.) *
β 11641	AC	18·74	6·0, 9·6	176·73	222·0	2	Position angle erratic. *
,,	BC	18·74	8·2, 9·6	174·60	206·90	2	*
β 11761	AC	18·64	9·2, 9·0	59·33	52·66	2	(Krüger 60.) (P.M.)
,,	AD	18·64	9·2, 8·0	140·27	201·54	2	(P.M.)
β 11968	AB	18·79	7·0, 7·5	261·2	3·32	2	
,,	AC	18·79	7·0, 8·2	122·48	46·98	2	
β 12053		18·80	8·7, 10·5	87·37	...	2	
β 12104		18·94	8·4, 8·5	112·3	...	1	
β 12179		18·74	5·6, 10·6	348·14	48·28	2 (1)	Slow relative motion.
β 12214	AB–C	18·62	8·0, 9·4	327·92	40·99	1	(P.M.)
,,	AB–D	18·62	8·0, 10·0	280·10	50·39	1	(P.M.)
β 12290	AC	18·83	8·2, 10·8	358·93	104·96	2	(P.M.) Many older measures.
,,	AD	18·79	8·2, 9·5	330·15	192 32	2	(P.M.)
,,	AE	18·79	8·2, 9·5	294·82	205·67	2	No other measures.
12 Andromed.		18·78	6·0, 8·7	255·93	121·15	2	(P.M.)
β 12381		18·76	8·1, 10·4	278·9	7·84	1	(P.M.)
β 12467		18·71	9·5, 9·6	155·0	11·60	1	(P.M.)

Name.	Epoch. 1900+	Mag.	P.A.	D.	*n.*	Remarks.
β 12495	18·76	8·3, 8·5	119°·4	5″·28	2	Distance decreasing?
P. XXIII. 164 AB	18·66	7·0, 9·5	213·28	106·26	2	(P.M.) *
,, AC	18·66	7·0, 9·5	141·43	118·91	2	(P.M.)
,, AD	18·66	7·0, 9·2	190·99	171·32	2	(P.M.)
,, AE	18·66	7·0, 9·0	125·80	159·64	2	(P.M.)
,, AF	18·66	7·0, 9·0	191·95	225·28	2	(P.M.)
β 12538	18·65	8·9, 9·0	82·77	50·50	2 (1)	(P.M.) Large relative motion. *
β 12608 AC	18·76	6·8, 9·0	132·73	160·93	2	C=B.D. 74°·1049. *
β 12656 AC	18·76	7·7, 10·0	245·72	45·46	1	(P.M.) *
β 12693 A*a*	18·73	6·3, 9·8	324·08	155·49	2	A=U Cassiop. *
,, B*b*	18·73	7·7, 9·9	129·63	134·42	2	*
β 12701 AC	18·81	5·9, 8·5	333·81	57·09	2	*Cf.* Gen. Cat., p. 1046.

Notes.

β 391. A and B have common p.m. (measures 1903–1919); perhaps also some relative motion in δ.

Lal. 2682. From AB and AD exactly same p.m. of A,

$$\mu_\alpha = +0''\cdot515,\ \mu_\delta = +0''\cdot213.$$

For meridian values, *cf.* (P.M.).

β 878. Measures (1900–1919) too discordant for reliable p.m.

B.D. 67°·191. p.m. from measures 1892–1919,

$$\mu_\alpha = +0''\cdot543,\ \mu_\delta = +0''\cdot290.$$

For meridian values, *cf.* (P.M.).

β 1520. Older measures:

1901·56	56°·4	3″·20	Kü. 2*n.*
1909·84	55°·1	3″·44	Doo. 2*n.*
1912·13	53°·8	2″·94	De 3*n.*

My measures disagree with those of Dearborn, which are termed erratic there. My separate results are:

62°·3 4″·01 and 56°·1 3″·55.

Lal. 5586. p.m. deduced from place in A.G. Lund. 1889, and measures of β (1907·8), and my own,

$$\mu_\alpha = +0''\cdot161,\ \mu_\delta = +0''\cdot030.$$

Br. 536. p.m. from four sets of measures (1886–1919),

$$\mu_\alpha = -0''\cdot015,\ \mu_\delta = -0''\cdot022.$$

Meridian values very discordant:

Boss $\mu_\alpha = -0''.023$, $\mu_\delta = +0''.013$.
Newcomb $\mu_\alpha = -0''.180$, $\mu_\delta = +0''.024$.

39 Tauri. p.m. from measures 1887–1918,

$\mu_\alpha = +0''.184$, $\mu_\delta = +0''.059$.

β 3423. From three sets of measures of B.C. (1880–1919), the p.m. of the close pair is found to be,

$\mu_\alpha = -0''.022$, $\mu_\delta = +0''.071$.

β 4159. The relative p.m. of this pair of equal magnitude from four sets of measures (1875–1919),

$\mu_\alpha = -0''.013$, $\mu_\delta = +0''.029$.

Lal. 15976. Relative motion of A from measures AB (1893–1919),

$\mu_\alpha = -0''.352$, $\mu_\delta = +0''.234$.

Relative motions of B from BC (1908–1919),

$\mu_\alpha = -0''.008$, $\mu_\delta = -0''.092$,

but distance measures discordant.

β 4609. From four sets (1888–1919),

$\mu_\alpha = -0''.130$, $\mu_\delta = +0''.110$,

agrees with meridian values, *cf.* (P.M.).

γ Cancri. p.m. from four sets (1888–1919),

$\mu_\alpha = -0''.104$, $\mu_\delta = +0''.026$,

agrees with meridian values, *cf.* (P.M.).

Pi. VIII. 174. p.m. of A is given from three sets of AB and AC (1893–1919), in good accordance, showing relative motions of *BC* to be imperceptible.

From AB is $\mu_\alpha = +0''.049$, and $\mu_\delta = +0''.208$.
From AC is $\mu_\alpha = +0''.055$, and $\mu_\delta = +0''.218$.

For meridian values, *cf.* (P.M.).

β 4819. p.m. from four sets (1875–1919),

$\mu_\alpha = +0''.051$, $\mu_\delta = +0''.025$.

Lal. 19627. p.m. from three sets (1893–1919),

$\mu_\alpha = -0''.234$, $\mu_\delta = +0''.096$.

Meridian values very discordant *cf.* (P.M.).

Gr. 1618. p.m. from three sets (1894–1919),

$\mu_\alpha = -1''.303$, $\mu_\delta = +0''.483$.

For meridian values *cf.* (P.M.).

Gr. 1636. p.m. from three sets (1893–1919),

$\mu_\alpha = +0''.010$, $\mu_\delta = +0''.257$.

All meridian values larger, *cf.* (P.M.).

β 5695. All the measures of this pair are:

1881·85	264°·3	36″·03
1894·37	264°·6	35″·59
1918·36	265°·6	35″·15

This pair has a common p.m. of 0″·546; β finds no relative change, but now it is evident, and a pair of the 61 Cygni type.

β 6043. Kü. No. 2. All the measures are:

1901·32	254°·2	8″·84	Kü.	(2)	Increase of angle,
1911·62	257°·1	9″·85	Doo.	(4)	change in distance
1912·07	256°·3	9″·16	De	(3)	not yet certain.
1919·24	258°·1	9″·40		(3)	

Lal. 22662 and 22667. Relative p.m. from three sets (1891–1919),

$$\mu_\alpha = -0''\cdot146,\ \mu_\delta = -0''\cdot592.$$

Küstner gives p.m. of A and B; from them the relative motion of A is:

$$\mu_\alpha = -0''\cdot176,\ \mu_\delta = -0''\cdot573.$$

β 6084. From four sets of AC (1864–1919),

$$\mu_\alpha = +0''\cdot090,\ \mu_\delta = +0''\cdot130.$$

Meridian p.m. has another direction, *cf.* (P.M.).

β 6084 AD. β gives $\Delta\delta = 136''\cdot7$ for 1901·1. These measures do not agree with the value of $\Delta\delta = 139''\cdot9$ in the A.G. Leipzig for 1884·1.

χ Virg. p.m. from four sets of AB (1886–1919),

$$\mu_\alpha = -0''\cdot050,\ \mu_\delta = +0''\cdot024.$$

Meridian observations give larger values, *cf.* (P.M.).

34 Virg. From two sets (1909–1919),

$$\mu_\alpha = +0''\cdot101,\ \mu_\delta = +0''\cdot062.$$

This does not agree with the meridian p.m. of A. [Auwers +0″·038, +0″·019; Boss +0″·048, +0″·026], but the interval is only ten years.

Pi. XII. 188. From three sets of AB (1893–1919),

$$\mu_\alpha = +0''\cdot212,\ \mu_\delta = +0''\cdot123.$$

For meridian values, *cf.* (P.M.).

Pi. XII. 188 AC. No change; measures Engelmann must have been erroneous.

β 6778. Increase in distance as announced in Gen. Cat. not confirmed by my measures and those of Dearborn.

1908·43	236°7	13″·12	(3*n*) De.

β 6837. Change in distance confirmed. Some of the latest measures are :

1907·31	169°·2	5″·59	Lau A. N. 4193.
1909·45	167°·6	5″·57	De (3*n*).

β 7259. Distance probably erroneous, though separate measures agree beautifully (1″·85 and 1·″73).

1904·61	68°·6	1″·06	(5*n*) Strbg.
1906·22	62°·1	1″·02	(2*n*) A.

β 7327. Relative p.m. from three sets (1903–1919)

$\mu_\alpha = -0''\cdot094$, $\mu_\delta = -0''\cdot076$.

Pair of equal magnitude.

ψ Serp. A*a*. $\mu_\alpha = -0''\cdot047$, $\mu_\delta = +0''\cdot135$.

(Boss, $\mu_\alpha = -0''\cdot086$, $\mu_\delta = +0''\cdot150$.)

β 7442. From four sets (1879–1919) (first epoch one night only),

$\mu_\alpha = +0''\cdot040$, $\mu_\delta = -0''\cdot037$.

Meridian p.m. larger, *cf.* (P.M.).

15 Herc. AB. From four sets (1887–1919),

$\mu_\alpha = -0''\cdot044$, $\mu_\delta = -0''\cdot065$.

Meridian p.m. discordant, *cf.* (P.M.).

β 7699. Both stars have a common p.m. of 0″·112 (Radcl, 1890). Perhaps some increase in distance detected (*cf.* Gen. Cat.), if so this is a pair of the 61 Cygni type.

30 Oph. From four sets, (1887–1919),

$\mu_\alpha = -0''\cdot040$, $\mu_\delta = +0''\cdot038$.

Does not agree with meridian p.m., *cf.* (P.M.), but the older measures are discordant.

β 7858 AC. From two sets (1907–1919),

$\mu_\alpha = -0''\cdot011$, $\mu_\delta = +0''\cdot051$.

For p.m. from AB, *cf.* (P.M.).

B.D. 42°·2810. The large motion, given in the *Bul. Ac. Sci. Petersbourg*, Febr. 1912, is real.

From two sets, (1912–1919),

$\mu_\alpha = -0''\cdot964$, $\mu_\delta = +0''\cdot516$.

B.D. 18°·3424. The proper motion of the principal star (1″·255 in 314°·5) is not confirmed by my measures of $\Delta\delta$ of AB and AC.

β 8295 AB. From five sets of measures, (1874–1919),

$\mu_\alpha = -0''\cdot046$, $\mu_\delta = -0''\cdot107$.

AB is an equal-magnitude pair (7·0 – 7·1).

β 8295 BC. The measures of BC prove that A is at rest. From two sets (1911–1919),

$$\mu_\alpha = +0''\cdot038, \quad \mu_\delta = +0''\cdot101.$$

β 8404 AB. From four sets (1868–1918),

$$\mu_\alpha = +0''\cdot005, \quad \mu_\delta = +0''\cdot033.$$

Oxf. 26°·55357. From photographs a large motion of the principal star (B.D. 25°·3780) was detected.

From two sets with a very short interval (1910–1919), I deduced

$$\mu_\alpha = +0''\cdot152, \quad \mu_\delta = -0''\cdot170.$$

β 9150. Probably fixed. Other measures are :

1881·46	271°·2	3″·30	(3*n*) β.
1887·80	269°·2	3″·01	(3*n*) Com.
1905·68	269°·4	2″·94	(1*n*) β.
1908·89	268°·3	3″·07	(3*n*) De.
1910·87	270°·5	2″·72	(3*n*) Doo.

β 9607 AD. p.m. small but real. From three sets (1880–1918),

$$\mu_\alpha = +0''\cdot009, \quad \mu_\delta = -0''\cdot025.$$

β 9652. No change in angle since 1907. The angle which β deduces from A.G. Leipzig is probably erroneous.

15 Sagittæ AB. The relative motion of AB from three sets (1886–1918) is

$$\mu_\alpha = -0''\cdot362, \quad \mu_\delta = +0''\cdot354.$$

15 Sagittæ AC. The relative motion of AC from three sets (1886–1918) is

$$\mu_\alpha = -0''\cdot384, \quad \mu_\delta = +0''\cdot390.$$

Therefore the motion of *C*, if *B* is at rest:

$$\mu_\alpha = +0''\cdot022, \quad \mu_\delta = -0''\cdot036.$$

15 Sagittæ B*b*. The hypothesis that B is at rest is confirmed by the measures of *Bb*, which show no perceptible change between 1908–1918.

29 Cygni AB. From three sets (1887–1918) the relative motion of A is

$$\mu_\alpha = +0''\cdot024, \quad \mu_\delta = -0''\cdot034.$$

29 Cygni AC. From four sets (1887–1918) the relative motion of A is

$$\mu_\alpha = +0''\cdot098, \quad \mu_\delta = -0''\cdot083.$$

For meridian motions of A and B, *cf.* (P.M.).

29 Cygni BD. From three sets (1894–1918) the relative motion of B is

$$\mu_\alpha = -0''\cdot065, \quad \mu_\delta = +0''\cdot040.$$

Cephei H 6. From three sets (1894–1918),

$$\mu_\alpha = -0''\cdot100,\ \mu_\delta = +0''\cdot253.$$

For meridian p.m., *cf.* (P.M.).

Lal. 40604. A and B have both different large proper motions. They have been both connected to the star *C*, which in its turn is connected to a faint star D. The relative motion of CD is very small, if not unreal. From two sets with an interval of ten years, $\mu_\alpha = +0''\cdot049$, $\mu_\delta = +0''\cdot033$; a measure of Engelmann on one night being rejected as erratic.

From three sets of AC (1894–1918), the p.m. of A is

$$\mu_\alpha = +0''\cdot261,\ \mu_\delta = -0''\cdot200.$$

From two sets of BC (1908–1918), the p.m. of B,

$$\mu_\alpha = -0''\cdot134,\ \mu_\delta = +0''\cdot073.$$

The direct measures of AB do not agree with these results.

B.D. 10°·4481. From three sets (1892–1918)

$$\mu_\alpha = +0''\cdot216,\ \mu_\delta = -0''\cdot043.$$

For meridian p.m., *cf.* (P.M.).

Oxf. 26°·74272. Large p.m. detected on photographs.

From two sets with very short interval (1910–1918),

$$\mu_\alpha = -0''\cdot333,\ \mu_\delta = +0''\cdot139.$$

Br. 2926. From four sets (1887–1918),

$$\mu_\alpha = +0''\cdot241,\ \mu_\delta = -0''\cdot124.$$

In excellent accordance with meridian p.m., *cf.* (P.M).

ϵ Cephei. From four sets (1887–1918),

$$\mu_\alpha = +0''\cdot460,\ \mu_\delta = -0''\cdot036.$$

In good accordance with meridian p.m.

β 11641 BC. From three sets (1880–1918) of BC,

$$\mu_\alpha = +0''\cdot050,\ \mu_\delta = -0''\cdot060.$$

12 Andr. β wrongly supposes that the difference between the meridian p.m. of A and that deduced from his measures is due to errors of measurement. B most certainly has a p.m. of its own.

From four sets (1886–1918),

$$\mu_\alpha = +0''\cdot079,\ \mu_\delta = -0''\cdot001.$$

Boss gives for A,

$$\mu_\alpha = +0''\cdot115,\ \mu_\delta = +0\cdot078.$$

Therefore absolute p.m. of B,

$$\mu_\alpha = +0''\cdot036,\ \mu_\delta = +0''\cdot079.$$

Pi. XXIII. 164. The p.m. of A is derived from five neighbouring stars, all measured at three epochs (1894–1918). The separate results agree well, and give no indications as to individual p.m. of the reference stars.

p.m. of A	μ_α	μ_δ	
From AB	$+0''\cdot373$	$-0''\cdot476$	mean
„ AC	$+0''\cdot356$	$-0''\cdot507$	$\mu_\alpha = +0''\cdot375$
„ AD	$+0''\cdot380$	$-0''\cdot473$	$\mu_\delta = -0''\cdot489$
„ AE	$+0''\cdot372$	$-0''\cdot497$	
„ AF	$+0''\cdot392$	$-0''\cdot493$	

β 12538. An interval of eight years, too short for deduction of p.m.

β 12608 AC. From four sets (1894–1918),

$$\mu_\alpha = +0''\cdot310,\ \mu_\delta = -0''\cdot055.$$

Boss gives for A,

$$\mu_\alpha = +0''\cdot329,\ \mu_\delta = -0''\cdot052.$$

β 12656 AC. My measures disagreeing with those of 1892–1909, no reliable p.m. can be deduced.

β 12693 A*a*. From three sets (1879–1918),

$$\mu_\alpha = -0''\cdot018,\ \mu_\delta = -0''\cdot018.$$

Nowhere are other values for the p.m. to be found.

β 12693 B*b*. From measures in 1879, 1909, and 1918, p.m. imperceptible.